Davide La Torre e Matteo Rocca

Approximating continuous functions by iterated function systems and optimization problems

2002/13

UNIVERSITÀ DELL'INSUBRIA
FACOLTÀ DI ECONOMIA

http://eco.uninsubria.it

In questi quaderni vengono pubblicati i lavori dei docenti della Facoltà di Economia dell'Università dell'Insubria. La pubblicazione di contributi di altri studiosi, che abbiano un rapporto didattico o scientifico stabile con la Facoltà, può essere proposta da un professore della Facoltà, dopo che il contributo sia stato discusso pubblicamente. Il nome del proponente è riportato in nota all'articolo. I punti di vista espressi nei quaderni della Facoltà di Economia riflettono unicamente le opinioni degli autori, e non rispecchiano necessariamente quelli della Facoltà di Economia dell'Università dell'Insubria.

These Working papers collect the work of the Faculty of Economics of the University of Insubria. The publication of work by other Authors can be proposed by a member of the Faculty, provided that the paper has been presented in public. The name of the proposer is reported in a footnote. The views expressed in the Working papers reflect the opinions of the Authors only, and not necessarily the ones of the Economics Faculty of the University of Insubria.

Printed in Italy in April 2002
Università degli Studi dell'Insubria
Via Ravasi 2, 21100 Varese, Italy

Approximating continuous functions by iterated function systems and optimization problems*

Davide La Torre† Matteo Rocca‡

22nd April 2002

Abstract

In this paper some new contractive operators on $C([a,b])$ of IFS type are built. Inverse problems are introduced and studied by convex optimization problems. A stability result and some optimality conditions are given.

AMS Subject Classification: 28A80, 41A20
Keywords: Iterated function systems, optimization

1 Introduction

The Iterated Function Systems (IFS) were born in the eighties ([2] [7]) as applications of the theory of discrete dynamical systems and as useful tools to build fractals and other similar sets. Some possible applications of IFS can be found in image processing theory (see [4] [6] and the references therein), in the theory of stochastic growth models (see [9] and the references therein) and in the theory of random dynamical systems ([1] [5] [8]). The fundamental result on which the IFS method is based is Banach theorem. In practical applications, the following *inverse problem* is fundamental: given f in a complete metric space (X, d), find a contractive operator $T : X \to X$ that admits a unique fixed point $\tilde{f} \in X$ such that $d(f, \tilde{f})$ is small enough. In fact if one is able to solve the inverse problem with arbitrary precision, it is possible to identify f with the operator T which has it as fixed point. In this paper, we analyze some contractive operators on $C([a,b])$ which arise as generalizations of adjoint operators associated to invariant measures of IFS and we show how the inverse problems can be formulated as constrained convex optimization problems.

*This work has been supportted by the F.A.R. 2001, of the University of Insubria.

†Contact author: University of Milan, Department of Economics, Faculty of Political Sciences, via Conservatorio,7, 20122, Milano, Italy. Tel.: +39276074462. Fax: +39276023198 E-mail: davide.latorre@unimi.it

‡University of Insubria, Department of Economics, Faculty of Economics, via Ravasi, 2, 21100, Varese, Italy. Tel: +39258365693. Fax: +39258365617. E-mail: mrocca@eco.uninsubria.it

The paper is structured as follows: section 2 is devoted to recall the principal notions of the theory of IFS; in section 3 and 4 two new contractive operators are introduced and the inverse problems are studied. Finally in section 5 a stability result is given.

2 Contractive maps on $C([a,b])$: an introduction

The aim of this section is to show how a class of contractions on $C([a,b])$ arise as adjoint operators of iterated function systems. If we consider the space $H([a,b])$, built with all compact subsets of $[a,b]$, and if we introduce on $H([a,b])$ the following metric (Hausdorff metric):

$$h(A,B) = \max\{\max_{x\in A}\min_{y\in B}|x-y|, \max_{x\in B}\min_{y\in A}|x-y|\}$$

then the space $(H([a,b]),h)$ is a complete metric space ([7]). Now let $B([a,b])$ denote the σ-algebra of Borel subsets of $[a,b]$ and $M([a,b])$ denote the space of all probability measures on $[a,b]$. Define a metric on $M([a,b])$ as:

$$d_H(\mu,\nu) = \sup_{f\in L_1} \int_a^b f d\mu - \int_a^b f d\nu, \quad \mu,\nu \in M([a,b])$$

where $L_1 = \{f : [a,b] \to \mathbb{R} : |f(x)-f(y)| \leq |x-y|\}$. Then $(M([a,b]),d_H)$ is a complete metric space [7] (this metric is called Hutchinson metric). Now let $w = \{w_1, w_2, \dots w_n\}$ denote a set of n continuous contraction maps on $[a,b]$, i.e. $w_i : [a,b] \to [a,b]$ and:

$$|w_i(x) - w_i(y)| \leq c_i|x-y|, \quad x,y \in [a,b],\ 0 \leq c_i < 1,\ i = 1\dots n.$$

The couple $([a,b],w)$ is called Iterated Function Systems (IFS). It will be convenient to define the maximum contractivity factor of the IFS as:

$$c = \max_{i=1\dots n} c_i < 1.$$

Associated with these maps is a set of non-zero probabilities, $p = \{p_1, p_2, \dots p_n\}$, $p_i > 0$ and $\sum_{i=1}^n p_i = 1$. Now for a set $S \in H([a,b])$, denote $w_i(S) = \{w_i(x), x \in S\}$ and denote the "parallel" action of the set of maps w_i on S as:

$$w(S) = \bigcup_{i=1}^n w_i(S).$$

Also define the iteration sequence $w^{s+1}(S) = w(w^s(S))$, $s = 1, 2 \dots$. Two important results for contractive IFS are given below.

Theorem 2.1. *[6]*

i) There exists a unique compact subset $A \in H([a,b])$, the attractor of the IFS $\{[a,b], w\}$ (independent of p) such that:

$$A = w(A) = \bigcup_{i=1}^{n} w_i(A)$$

and $h(w^s(S), A) \to 0$ as $s \to \infty$ for all $S \in H([a,b])$.

ii) Define the following "Markov operator" $M : M([a,b]) \to M([a,b])$,

$$M(\nu) = \sum_{i=1}^{n} p_i \nu \circ w_i^{-1}.$$

Then there exists a unique measure $\mu \in M([a,b])$, termed the invariant measure, which obeys the fixed point condition:

$$M\mu = \mu.$$

Moreover, supp(μ) $= A$.

Note that the equation $A = w(A)$ implies that A is self-tiling, i.e. A is union of (distorted) copies of itself. This establish the geometric nature of the IFS. The equation $M\mu = \mu$ characterizes IFS in term of measures. Then for a μ-integrable function $f : [a,b] \to \mathbb{R}$,

$$\int_A f(x) d\mu(x) = \sum_{i=1}^{n} p_i \int_A f(w_i(x)) d\mu(x).$$

Let $C([a,b])$ the Banach space of continuous functions on $[a,b]$, and associated with the IFS $\{w,p\}$ define the following operator $T : C([a,b]) \to C([a,b])$:

$$Tf = \sum_{i=1}^{n} p_i f \circ w_i, \;\; f \in C([a,b]).$$

Now for a given $\mu \in M([a,b])$, define the linear functional $F : C([a,b]) \to \mathbb{R}$,

$$F(f) =< f, \nu >= \int_{[a,b]} f d\nu.$$

Then:

$$< Tf, \nu >=< f, M\nu >$$

i.e. T is the adjoint operator of M.

The operator T is a contraction on the complete metric space $(C([a,b]), d_\infty$[1]$)$ with contractivity factor $c = \max_{i=1 \ldots n} p_i < 1$. Now let $f \in C([a,b])$; then the inverse problem consists of finding an operator $T : C([a,b]) \to C([a,b])$, $Tf = \sum_{i=1}^{n} p_i f \circ w_i$, such that $Tf = f$. In approximating way this problem can be solved using the following result (known as Collage theorem).

[1] $d_\infty(f,g) = \max_{x \in [a,b]} |f(x) - g(x)|$

Theorem 2.2. *[6] Let $f \in C([a,b])$. If $d(Tf,f) \leq \epsilon$ then $d(f,\tilde{f}) \leq \frac{\epsilon}{1-c}$, where $T\tilde{f} = \tilde{f}$ and $c = \max_{i=1\ldots n} p_i < 1$.*

So the inverse problem is reduced to the problem of minimizing the quantity $d(Tf,f)$ over a functional space built with the maps w_i and the probabilities p_i. So if a family of maps w_i with associated probabilities p_i is given, as in [6], the inverse problem can be reduced to a convex constrained optimization problem on $\mathbb{R}^n$. The aim of the next sections is to introduce some generalizations of the map T and to study the inverse problems.

3 A new contractive operator of IFS type on $C([a,b])$

Now let us consider the function $T_1 : C([a,b]) \to C([a,b])$, which is a generalization of the operator T, defined as:

$$T_1 u(x) = \sum_{i=1}^{n} p_i \phi_i \circ u \circ w_i(x)$$

where $\phi_i : \mathbb{R} \to \mathbb{R}$ are Lipschitzian with Lipschitz constants $K_i \in \mathbb{R}_+$, $w_i : [a,b] \to [a,b]$ are continuous and $p_i \in \mathbb{R}_+$, $i = 1 \ldots n$. We have the following result which establishes that, under some hypotheses, T_1 is a contraction on $C([a,b])$. We observe that T_1 is reduced to T when $\phi_i(x) = x$, $\forall i = 1 \ldots n$.

Theorem 3.1. *$T_1 : C([a,b]) \to C([a,b])$ verifies the following inequality:*

$$d_\infty(T_1 f, T_1 g) \leq \left\{ \sum_{i=1}^{n} p_i K_i \right\} d_\infty(f,g).$$

Proof. In fact we have $\forall f,g \in C([a,b])$:

$$d_\infty(T_1 f, T_1 g) = \max_{x\in[a,b]} |T_1 f(x) - T_1 g(x)| \leq$$

$$\max_{x\in[a,b]} \sum_{i=1}^{n} p_i \, |\phi_i(f(w_i(x))) - \phi_i(g(w_i(x)))| \leq$$

$$\max_{x\in[a,b]} \sum_{i=1}^{n} p_i K_i \, |f(w_i(x)) - g(w_i(x))| \leq$$

$$\left\{ \sum_{i=1}^{n} p_i K_i \right\} d_\infty(f,g).$$

□

In this context, Banach theorem ensures that if $K_{T_1} := \sum_{i=1}^{n} p_i K_i$ is less than one then the map T_1 is a contraction on $C([a,b])$; so the fixed point equation $T_1 f = f$ has a unique solution $\tilde{f}$ and the sequence $T_1^j u = T_1(T_1^{j-1} u)$ converges to $\tilde{f}$ for all

$u \in C([a,b])$. Given a function $f \in C([a,b])$ the inverse problem consists of finding a contractive operator T_1, as above, such that the fixed point of T_1 is near to f. If the Lipschitzian maps ϕ_i and the continuous functions w_i are given, the solutions of the inverse problem have to be found by using the parameters p_i (we will write $T_{1,p}$ instead of T_1 to put in evidence the dependence of the operator T_1 on the vector $p = (p_1, p_2, \dots, p_n)$). Furthermore, since the fixed point $\tilde{f}_p$ of a map $T_{1,p}$ is unknown, is useful the following theorem which gives us an estimate of $\tilde{f}_p$.

Theorem 3.2. *Let $p^* \in \mathbb{R}^n_+$ such that $K_{T_1} = \sum_{i=1}^n p_i^* K_i < 1$ and $f \in C([a,b])$. If $\tilde{f}_{p^*}$ is the unique fixed point of T_{1,p^*} and if $d_\infty(f, T_{1,p^*} f) < \epsilon$ then:*

$$d_\infty(f, \tilde{f}_{p^*}) < \frac{\epsilon}{1 - K_{T_1}}.$$

Proof. In fact:

$$d_\infty(f, \tilde{f}_{p^*}) \le d_\infty(f, T_{1,p^*} f) + d_\infty(\tilde{f}_{p^*}, T_{1,p^*} \tilde{f}_{p^*}) + d_\infty(T_{1,p^*} \tilde{f}_{p^*}, T_{1,p^*} f)$$

and, since $T_{1,p^*} \tilde{f}_{p^*} = \tilde{f}_{p^*}$, we obtain:

$$d_\infty(f, \tilde{f}_{p^*}) \le \epsilon + K_{T_1} d_\infty(f, \tilde{f}_{p^*})$$

that is the thesis. □

In other words, the previous theorem states that if f is given and p^* is such that $d_\infty(T_{1,p^*} f, f) < \epsilon$ (with ϵ "sufficiently small") then, $\forall u \in C([a,b])$, the sequence $T^s_{1,p^*} u = T_{1,p^*}(T^{s-1}_{1,p^*} u)$ converges to $\tilde{f}_{p^*}$ (when $s \to +\infty$) which is an approximation of f. Obviously now the problem consists of finding $p^* \in \mathbb{R}^n_+$ such that:

$$P1) \qquad F(p^*) = \min F(p) := \min d_\infty(T_{1,p} f, f)$$

with $\sum_{i=1}^n p_i K_i \le \delta$ and $\delta < 1$. We will see that this is a constrained convex optimization problem. It is clear that the ideal solution consists of finding p^* such that $F(p^*) = 0$; in fact in this case the map $T_{1,p}$ has exactly f as fixed point. Since F is in general not differentiable, this is a nonsmooth equation. Furthermore F is convex and then semismooth; this assures the convergence of the Newton's method introduced by Qi and Sun [10]. In the other cases the value $F(p^*)$ is a lower bound for the precision of the estimate; in this setting this bound can be improved only increasing the number n of parameters p_i.

Remark 3.1. *If there exists a vector $p \in \mathbb{R}^n$ such that $T_{1,p} f = f$ then f is a solution of the following functional equation:*

$$f = \sum_{i=1}^k p_i \phi_i \circ f \circ w_i.$$

With the following results we study the previous minimization problem and we give a necessary and sufficient condition for the existence of a minimizer.

Theorem 3.3. $F : \mathbb{R}_+^n \to \mathbb{R}_+$ *is convex.*

Proof. $\forall p_1, p_2 \in \mathbb{R}_+^n$ and $\forall t \in [0,1]$ we have:

$$F(tp_1 + (1-t)p_2) = \max_{x\in[a,b]} \left| \sum_{i=1}^{n} (tp_{1,i} + (1-t)p_{2,i})\phi_i(f(w_i(x))) - f(x) \right| \leq$$

$$tF(p_1) + (1-t)F(p_2).$$

$\square$

Proposition 3.1. *The set* $C = \{p \in \mathbb{R}_+^n : \sum_{i=1}^n p_i K_i \leq \delta\}$ *is convex.*

So the problem $P1)$ is a constrained convex optimization problem. A necessary and sufficient condition for the existence of minimizer, given by subdifferential theory ([11]), is given in the following result.

Theorem 3.4. $p^* \in C$ *is a minimizer of* $F(p)$ *if and only if there exist* $\lambda_i \geq 0$, $i = 1 \ldots n+1$ *such that* p^* *is a solution of the following system:*

$$\begin{cases} 0 \in \partial_p\{F(p) - \sum_{i=1}^n \lambda_i p_i + \lambda_{n+1}(\sum_{i=1}^n p_i K_i - \delta)\} \\ \\ \lambda_i \geq 0, \ i = 1 \ldots n+1 \\ \sum_{i=1}^n p_i K_i - \delta \leq 0, \\ \lambda_i p_i = 0, -p_i \leq 0, i = 1 \ldots n \\ \lambda_{n+1}(\sum_{i=1}^n p_i K_i - \delta) = 0 \end{cases}$$

4 A definitively contractive operator of IFS type on $C([a,b])$

The aim of this section is to show how, changing the family of IFS maps $T_{1,p}$, one can weaken the requests on the constraints in $P1)$. To do this, let us consider on $(C([a,b]), d_\infty)$ the following modified map $T_{2,p}$:

$$T_{2,p}f(x) = \int_a^x \sum_{i=1}^n p_i \phi_i \circ f \circ w_i(t) dt + c$$

where $\phi : \mathbb{R} \to \mathbb{R}$ are Lipschitzian functions with Lipschitz constants $K_i \geq 0$, $w_i : [a,b] \to [a,b]$ are continuous, $c \in \mathbb{R}$. With these assumptions it is trivial to prove that $T_{2,p} : C([a,b]) \to C([a,b])$. Before analyzing the contractivity of $T_{2,p}$ let us to recall the following classical definition.

Definition 4.1. *Let* (X,d) *be a metric space. A function* $T : X \to X$ *is said to be definitively contractive on* X *if there exists* $m \in \mathbb{N}$ *and a positive constant* $K < 1$ *such that:*

$$d(T^m f, T^m g) \leq K d(f,g)$$

where $T^m f = T(T^{m-1} f))$ *and* $T^0 f = f$.

It is well known that for definitively contractive operators the following generalized version of Banach theorem holds.

Theorem 4.1. *Let (X, d) be a complete metric space and $T : X \to X$ be a definitively contractive operator on X. Then there exists a unique element $\tilde{f} \in X$ such that*

$$\begin{cases} T\tilde{f} = \tilde{f}, \\ \\ \lim_{s\to+\infty} d(T^s u, \tilde{f}) = 0, \ \forall u \in X. \end{cases}$$

We are ready to prove the following fundamental result.

Theorem 4.2. *$T_{2,p} : C([a, b]) \to C([a, b])$ is a definitively contractive map on $C([a, b])$.*

Proof. With trivial calculations one can deduce that:

$$|T_{2,p}^m f(x) - T_{2,p}^m g(x)| \leq \frac{(x-a)^m (\sum_{i=1}^n p_i K_i)^m}{m!} d_\infty(f, g)$$

and then:

$$d_\infty(T_{2,p}^m f, T_{2,p}^m g) \leq \frac{(b-a)^m (\sum_{i=1}^n p_i K_i)^m}{m!} d_\infty(f, g) := K_m d_\infty(f, g).$$

Since $\lim_{m\to+\infty} K_m = 0$ then T is a definitively contractive map when m is big enough. □

Remark 4.1. *The function $T_{2,p}u$ is differentiable even if u is only continuous.*

The inverse problem for the map $T_{2,p}$ can be written by using a result similar to theorem 3.2.

Theorem 4.3. *Let $f \in C([a, b])$ and $p \in \mathbb{R}^n_+$. If $d_\infty(T_{2,p} f, f) < \epsilon$ then $\exists K \in \mathbb{R}_+$ such that*

$$d_\infty(\tilde{f}_p, f) < K\epsilon,$$

where $\tilde{f}_p$ is the fixed point of $T_{2,p}$.

Proof. Since $T_{2,p}$ is a definitively contractive map with Lipschitz constant K_1, then there exists $m_0 \in \mathbb{N}$ such that $T_{2,p}^{m_0}$ is a contraction with Lipschitz constant $K_2 < 1$ and $\tilde{f}_p$ is its fixed point. So we have:

$$d(T_{2,p}^{m_0} f, f) \leq \sum_{i=0}^{m_0-1} d(T_{2,p}^{m_0-i} f, T_{2,p}^{m_0-i-1} f) \leq \sum_{i=0}^{m_0-1} K_1^{m_0-i-1} d(T_{2,p} f, f) \leq C\epsilon.$$

Now we can apply theorem 3.2 to the function $T_{2,p}^{m_0}$, obtaining:

$$d(\tilde{f}_p, f) \leq \frac{C\epsilon}{1 - K_2} := K\epsilon.$$

□

Remark 4.2. *If there exists a vector* $p \in \mathbb{R}^n_+$ *such that* $T_{2,p}f = f$ *then* f *is a solution of the following integral equation:*

$$f(x) = \int_a^x \sum_{i=1}^n p_i \phi_i \circ f \circ w_i(t) dt + f(a).$$

One can observe that the solution of the previous functional equations has to be a differentiable function. So, fixed a positive continuous function f*, the inverse problem can be exactly solved only when* f *is differentiable; in the other cases only an approximate solution can be given. Furthermore* f *is a solution of the following ordinary differential equation:*

$$f'(x) = \sum_{i=1}^n p_i \phi_i \circ f \circ w_i(x).$$

Choosing $c = f(a)$, the inverse problem is reduced to the solution of the following constrained optimization problem:

$$\min_{p \in \mathbb{R}^n_+} d_\infty(T_{2,p}f, f) := \min_{p \in \mathbb{R}^n_+} F(p).$$

Even if the best solution of the previous problem is to find $p^* \in \mathbb{R}^n_+$ such that $F(p^*) = 0$ (in fact in this case we should found an operator T which has exactly f as fixed point), the previuos theorem states that if $F(p^*)$ is small enough then the map T_{2,p^*} has a fixed point $\tilde{f}_{p^*}$ near to f. We now analyze some properties of the function $F : \mathbb{R}^n_+ \to \mathbb{R}_+$.

Theorem 4.4. $F : \mathbb{R}^n \to \mathbb{R}_+$ *is convex.*

Proof. Choose $p_1, p_2 \in \mathbb{R}^n$ and $t \in [0, 1]$. Then:

$$F(tp_1 + (1-t)p_2) =$$

$$\max_{x \in [a,b]} \left| \int_{[a,x]} \sum_{i=1}^n (tp_{1,i} + (1-t)p_{2,i}) \phi_i \circ f \circ w_i(t) dt + c - f(x) \right| \leq$$

$$tF(p_1) + (1-t)F(p_2).$$

$\square$

A necessary and sufficient condition for a $p \in \mathbb{R}^n_+$ to be a minimizer of F is shown in the next result.

Theorem 4.5. $p^* \in \mathbb{R}^n_+$ *is a minimizer of* $F(p)$ *if and only if there exist* $\lambda_i \geq 0$, $i = 1 \ldots n$ *such that* p^* *is a solution of the following system:*

$$\begin{cases} 0 \in \partial_p \{F(p) - \sum_{i=1}^n \lambda_i p_i\} \\ \\ \lambda_i \geq 0, \;\; i = 1 \ldots n \\ \lambda_i p_i = 0, -p_i \leq 0, i = 1 \ldots n \end{cases}$$

5 Convergence of minimizers and applications to optimality

In this section we will suppose that, given a continuous function f on $[a, b]$ and a map $T_p : C([a, b]) \to C([a, b])$, we are able to calculate a vector $p^* \in \mathbb{R}^n_+$ such that $F(p^*) = 0$, that is $T_{p^*} f = f$. We now show a result which can be used to give optimality conditions on f by using a sequence of approximating functions. In fact the calculus of minimizers of $T^n u$ can be easier and the following result guarantees the convergence to a minimizer of f.

Theorem 5.1. *If x_m, $m \in \mathbb{N}$, is the sequence of minimizers of $T_p^m u$, $\forall u \in C([a, b])$, then there exists a subsequence x_{m_k}, $k \in \mathbb{N}$, which converges to $x^* \in [a, b]$ and this point is a minimizer for f.*

Proof. Since $T_p^m u : [a, b] \to \mathbb{R}_+$ is a continuous function $\forall m \in \mathbb{N}$, then there exists a sequence of minimizers $\{x_m\}_{m\in\mathbb{N}}$. The sequence $\{x_m\}_{m\in\mathbb{N}}$ is a subset of the compact $[a, b]$ and so we can extract a subsequence $\{x_{m_k}\}_{k\in\mathbb{N}}$ which converges to $x^* \in [a, b]$. Our thesis is to show that x^* is an absolute minimizer of f. By using the uniform convergence of $T_p^m u$, we have that $\forall \epsilon > 0$, $\exists m_0(\epsilon)$ such that $\forall m \geq m_0(\epsilon)$, $\max_{x\in[a,b]} |T_p^m u(x) - f(x)| < \epsilon$. Now if $k \geq k_0(\epsilon)$ then $\max_{x\in[a,b]} |T_p^{m_k} u(x) - f(x)| < \epsilon$. In particular, choosing x_{m_k}, we have that $|T_p^{m_k} u(x_{m_k}) - f(x_{m_k})| \to 0, \quad k \to +\infty$. So we have:

$$|T_p^{m_k} u(x_{m_k}) - f(x^*)| \leq |T_p^{m_k} u(x_{m_k}) - f(x_{m_k})| + |f(x_{m_k}) - f(x^*)| \to 0$$

and

$$f(x^*) = lim_{k\to+\infty} T_p^{m_k} u(x_{m_k}) \leq lim_{k\to+\infty} T_p^{m_k} u(x) = f(x).$$

□

References

[1] Arnold L., Crauel H.: *Iterated function systems and multiplicative ergodic theory*, Diffusion processes and related problems in analysis (Pinsky M.A., Vihstutz V. eds.), vol. II, Stochastics flows, Progress in probability 27, Birkhuser, Boston, 1992, 283-305.

[2] Barnsley M.F., Demko S.: *Iterated function systems and the global construction of fractals*, Proc. Roy. Soc. London Ser.A, 399, 1985, 243-275.

[3] Barnsley M.F., Sloan A.D.: *A better way to compress images*, BYTE Magazine, January issue, 1988, 215-223.

[4] Crilly A.J., Earnshaw R.A., Jones H.: *Applications of fractals and chaos*, Springer Verlag, New York, 1993.

[5] Elton J.H., Piccioni M.: *Iterated function systems arising from recursive estimation problems*, Probab. Theory Relat. Fields, 91, 1992, 103-114.

[6] Forte B., Vrscay E.R.: *Theory of generalized fractal transforms*, Fractal Image Encoding and Analysis (Y.Fisher ed.), Springer Verlag, New York, 1996.

[7] Hutchinson J.,: *Fractals and self-similarity*, Indiana Univ. J. Math., 30, 1981, 713-747.

[8] Kwiecinska A.A., Slomczynski W.: *Random dynamical systems arising from iterated function systems with place-dependent probabilities*, Stat. and Prob. Letters, 50, 2000, 401-407.

[9] Montrucchio L., Privileggi F.: *Fractal steady states in stochastic optimal control models*, Annals of Operation Research, 88, 1999, 183-197.

[10] Qi L., Sun W.: *A nonsmooth version of Newton's method.* Math. Program., 58, 1993, 353-367.

[11] Rockafellar R.T., Wets R.J-B.: *Variational analysis*, Springer, 1998.