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A survey on $C^{1,1}$ functions: theory, numerical methods and applications*

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Abstract

In this paper we survey some notions of generalized derivative for $C^{1,1}$ functions. Furthermore some optimality conditions and numerical methods for nonlinear minimization problems involving $C^{1,1}$ data are studied.

MSC 2000: 26A24, 26A16

1 Introduction

Characterizing the optimal solutions by means of second order conditions is a problem of continuous interest in the theory of mathematical programming problems with twice continuously differentiable data. Recently, more attention has been paid to problems which don't involve C^2 data. One possible way is to reduce C^2 regularity assumptions to $C^{1,1}$ regularity (in the sense of the following definition).

Definition 1.1. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be of class $C^{1,1}$, or briefly a $C^{1,1}$ function, when f is differentiable and ∇f is locally Lipschitzian.

The class of $C^{1,1}$ functions was first brought to attention by Hiriart-Urruty in his doctoral thesis [20] and studied by Hiriart-Urruty J.B., Strodiot J.J., Hien Nguyen V. in [21]. The need for investigating such functions, as pointed out in [21, 23], comes from the fact that several problems of applied mathematics including variational inequalities, semi-infinite programming, penalty functions, augmented lagrangian, proximal point methods, iterated local minimization by decomposition etc. involve differentiable functions with no hope of being twice differentiable. In the following some examples of problems involving $C^{1,1}$ data are shown.

Example 1.1. Let $g : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable on Ω and consider[1] $f(x) = [g^+(x)]^2$ where $g^+(x) = \max\{g(x), 0\}$. Then f is $C^{1,1}$ on Ω.

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[1] This type of functions arises in some penalty methods.

Example 1.2. In many problems in engineering applications and control theory one has to study nonsmooth semi-infinite optimization problems as the following:

$$\text{minimize } f(x)$$

$$\text{subject to } \max_{t\in[a,b]} \phi_j(x,t) \leq 0,\ j = 1 \ldots l$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is C^2 and $\phi_j : \mathbb{R}^n \to \mathbb{R}$ is C^2, $j = 1 \ldots l$, $-\infty < a < x < b < +\infty$. One approach for solving this problem is to convert the functional constraints into equality constraints of the form:

$$h_j(x) = \int_a^b [\max\{\phi_j(x,y), 0\}]^2 \, dt = 0, j = 1 \ldots l$$

and apply the methods of nonlinear programming. Hence the problem becomes:

$$\text{minimize } f(x)$$

$$\text{subject to } h_j(x) = 0,\ j = 1 \ldots l.$$

Since ϕ_j is C^2, it is easy see that the function h_j is $C^{1,1}$ with the gradient:

$$\nabla h_j(x) = 2 \int_a^b \max\{\phi_j(x,t), 0\} \nabla \phi_j(x,t) dt,\ j = 1 \ldots l.$$

Example 1.3. Consider the following minimization problem:

$$\min f_0(x)$$

over all $x \in \mathbb{R}^n$ such that $f_1(x) \leq 0, \ldots f_m(x) \leq 0$. Letting r denote a positive parameter, the augmented Lagrangian L_r (see [45] and references therein) is defined on $\mathbb{R}^n \times \mathbb{R}^m$ as:

$$L_r(x,y) = f_0(x) + \frac{1}{4r} \sum_{i=1}^{m} \{[y_i + 2rf_i(x)]^+\}^2 - y_i^2.$$

From the general theory of duality which yields L_r as a particular Lagrangian, we know that $L_r(x,\cdot)$ is concave and also that $L_r(\cdot,y)$ is convex whenever the minimization problem is a convex minimization problem. By stating $y = 0$ in the previous expression, we observe that:

$$L_r(x,0) = f_0(x) + r \sum_{i=1}^{m} [f_i^+(x)]^2$$

is the ordinary penalized version of the minimization problem. L_r is differentiable everywhere on $\mathbb{R}^n \times \mathbb{R}^m$ with:

$$\nabla_x L_r(x,y) = \nabla f_0(x) + \sum_{j=1}^{m} [y_j + 2rf_j(x)]^+ \nabla f_j(x),$$

$$\frac{\partial L_r}{\partial y_i}(x,y) = \max\{f_i(x), -\frac{y_i}{2r}\}, i = 1 \dots m.$$

When the f_i are C^2 on $\mathbb{R}^n$, L_r is $C^{1,1}$ on $\mathbb{R}^{n+m}$. The dual problem corresponding to L_r is by definition:

$$\max g_r(y)$$

over $y \in \mathbb{R}^m$, where $g_r(y) = \inf_{x\in\mathbb{R}^n} L_r(x,y)$. In the convex case with $r > 0$, g_r is again $C^{1,1}$ concave function with the following uniform Lipschitz property on ∇g:

$$|\nabla g_r(y) - \nabla g_r(x)| \leq \frac{1}{2r}|y - y'|, \ \forall y, y' \in \mathbb{R}^m.$$

In [29] the following characterization of $C^{1,1}$ functions by divided differences is proved.

Theorem 1.1. *[31] Assume that the function $f : \Omega \to \mathbb{R}$ is bounded on a neighborhood of the point $x_0 \in \Omega$. Then f is of class $C^{1,1}$ at x_0 if and only if there exist neighborhoods U of x_0 and V of $0 \in \mathbb{R}$ such that $\frac{\Delta_2^d f(x;t)}{t^2}$ is bounded on $U \times V \backslash \{0\}$, $\forall d \in S^1 = \{d \in \mathbb{R}^n : \|d\| = 1\}$ where*

$$\Delta_2^d f(x;t) = f(x + 2td) - 2f(x + td) + f(x).$$

Remark 1.1. A similar result can be proved by using the following divided differences:

$$\delta_2^d f(x;t) = f(x + td) - 2f(x) + f(x - td).$$

It is known [55] that if a function f is of class $C^{1,1}$ at x_0 then it can be expressed (in a neighborhood of x_0) as difference of two convex functions. The following corollary strenghtens the results in [55].

Corollary 1.1. *[31] If f is of class $C^{1,1}$, then $f = \tilde{f} + p$ where $\tilde{f}$ is convex and p is a polynomial of degree at most two.*

2 Second order generalized derivatives for $C^{1,1}$ functions

Many second order generalized derivatives have been introduced to obtain optimality conditions for optimization problems with $C^{1,1}$ data. We will focus our attention on the definitions due to Hiriart-Urruty [20], Liu [34, 35, 36], Yang-Jeyakumar [57], Peano [44], Riemann [46]. Some of these definitions do not require the hypothesis of $C^{1,1}$ regularity; however, under this assumption, each derivative in the previous list is bounded.

The definitions of Hiriart-Urruty and Yang-Jeyakumar extend to the second order, respectively, the notions due to Clarke and Michel-Penot for the first order. Peano and Riemann definitions are classical ones. Peano introduced his definition while he was studying Taylor expansion formula for real functions. Peano derivatives were studied and generalized in recent years by Ben-Tal and Zowe [2] and Liu, who also obtained optimality conditions. Riemann higher-order derivatives were introduced in the theory of trigonometric series. Furthermore they were developed by several authors (for instance De la Vallee-Poussin and Denjoy [11, 12]). Applications of these notions to optimization problems were also given by Ginchev, Guerraggio and Rocca [14, 15, 16, 18].

2.1 Clarke and Michel-Penot generalized derivatives

Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be a Lipschitzian function, with Lipschitz constant K, and Ω be an open subset of $\mathbb{R}^n$. This means that the quantity:

$$\frac{\Delta_1^d f(x,t)}{t} = \frac{f(x+td) - f(x)}{t}$$

is uniformly bounded with respect to $d \in S^1$ (the unit sphere in $\mathbb{R}^n$) by the constant K. For this type of functions, Clarke generalized directional derivative and Michel-Penot generalized directional derivatives are given, rispectively, by:

$$\overline{f}'_C(x;d) = \limsup_{x' \to x, t \downarrow 0} \frac{\Delta_1^d f(x',t)}{t}$$

$$\overline{f}'_M(x;d) = \sup_{z \in \mathbb{R}^n} \limsup_{t \downarrow 0} \frac{\Delta_1^d f(x+tz,t)}{t}.$$

Then it follows from the definitions that:

$$\overline{f}'_D(x;d) \leq \overline{f}'_M(x;d) \leq \overline{f}'_C(x;d)$$

where:

$$\overline{f}'_D(x;d) = \limsup_{t \downarrow 0^+} \frac{\Delta_1^d f(x;t)}{t},$$

is the upper Dini derivative. The associate generalized subdifferentials are given by:

$$\partial_C f(x) = \{x^* \in \mathbb{R}^n : f'_C(x,d) \geq < x^*, d >, \forall d \in \mathbb{R}^n\};$$

$$\partial_M f(x) = \{x^* \in \mathbb{R}^n : f'_M(x,d) \geq < x^*, d >, \forall d \in \mathbb{R}^n\}.$$

Then it follows from the definitions that:

$$\partial_M f(x) \subseteq \partial_C f(x)$$

and the above inequality and inclusion may hold strictly [41]. In fact if we consider the function $f(x) = x^2 \sin\left(\frac{1}{x}\right)$ we have $\partial_C f(0) = [-1, 1]$ and $\partial_M f(0) = \{0\}$. For properties of Clarke and Michel-Penot generalize derivatives we refer to [8, 41].

According to Rademacher's theorem, a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable almost everywhere (a.e.) in the sense of Lebesgue measure. Let Ω_f be the set on which f fails to be differentiable. Then:

$$\partial_C f(x) = co\{\lim \nabla f(x_i) : x_i \to x, x_i \notin \Omega_f\},$$

where co denotes the convex hull. That is if we consider any sequence $x_i \to x$ such that the sequence $\nabla f(x_i)$ converges, then the convex hull of all such limit points is $\partial_C f(x)$ (see [8]).

Now assume that f is of class $C^{1,1}$. In Cominetti and Correa [9], a generalized second order directional derivative of a $C^{1,1}$ function in the directions (u, v) is defined in the sense of Clarke as follows:

$$\overline{f}''_C(x; u, v) = \limsup_{y \to x, t \downarrow 0} \frac{< \nabla f(y + tu), v > - < \nabla f(y), v >}{t}$$

and the generalized Hessian of f at x defined as for each $u \in \mathbb{R}^n$,

$$\partial^2_C f(x)(u) = \{x^* \in \mathbb{R}^n : f''_C(x; u, v) \geq < x^*, v >, \forall v \in \mathbb{R}^n\}.$$

In the following theorem some properties of $\overline{f}''_C$ are listed.

Theorem 2.1. *[9]*

- *The map $(u, v) \to \overline{f}''_C(x; u, v)$ is symmetric ($\overline{f}''_C(x; u, v) = \overline{f}''_C(x; v, u)$) and bisublinear (sublinear on each variable separately).*

- *The map $x \to \overline{f}''_C(x; u, v)$ is upper semicontinuous at x for every (u, v) and the point-to-set map $x \to \partial^2 f(x)(u)$ is closed at x for each fixed u.*

- $\overline{f}''_C(x; u, -v) = \overline{f}''_C(x; -u, v) = \overline{-f}''_C(x; u, v)$.

In Yang and Jeyakumar [55] a generalized second order directional derivative of a $C^{1,1}$ function in the directions (u, v) is defined in the sense of Michel-Penot as follows:

$$\overline{f}''_M(x; u, v) = \sup_{z \in \mathbb{R}^n} \limsup_{t \downarrow 0} \frac{< \nabla f(x + tz + tu), v > - < \nabla f(x + tz), v >}{t}$$

while the generalized Hessian is:

$$\partial^2_M f(x)(u) = \{x^* \in \mathbb{R}^n : f''_M(x; u, v) \geq < x^*, v >, \forall v \in \mathbb{R}^n\}.$$

In the following result some properties of $\overline{f}''_M$ are listed.

Theorem 2.2. *[55]*

- *The function $\overline{f}''_M(x; u, v)$ is bi-sublinear.*

- $\overline{f}''_M(x; u, -v) = \overline{f}''_M(x; -u, v) = \overline{-f}''_M(x; u, v)$

It is easy to see that $\overline{f}''_D(x; u) \leq \overline{f}''_M(x; u, u) \leq \overline{f}''_C(x; u, u)$, where:

$$\overline{f}''_D(x; u) = \limsup_{t \downarrow 0^+} \frac{< \nabla f(x + tu), u > - < \nabla f(x), u >}{t}$$

and hence $\partial^2_M f(x)(u) \subseteq \partial^2_C f(x)(u)$. In the following example is shown that the inclusion may be strict.

Example 2.1. Define:

$$f(x) = \int_0^x t^2 \sin\left(\frac{1}{t}\right) dt, \ x \in \mathbb{R}.$$

The function f is differentiable everywhere on $\mathbb{R}$ and

$$f'(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Hence f is $C^{1,1}$ and f is twice differentiable on $\mathbb{R}$ but not of class C^2 and:

$$\overline{f}''_M(0;1,v) = 0, \overline{f}''_C(0;1,v) = |v|$$

$$\partial^2_M f(0)(1) = \{f''(0)\} = \{0\}, \partial^2_C f(0)(1) = [-1,1].$$

Furthermore the functions $(x,u) \to \partial^2_M f(x)(u)$ and $\overline{f}''_M(x;u,v)$ are not upper semicontinuous. In [56] is proved the following result which gives a condition for the upper semicontinuity.

Proposition 2.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be $C^{1,1}$ and let $x \in \mathbb{R}^n$. Then for each $(x,u) \in \mathbb{R}^n$ the function $y \to \overline{f}''_M(x;u,v)$ is upper semicontinuous at $x \in \mathbb{R}^n$ if and only if:*

$$\overline{f}''_M(x;u,v) = \overline{f}''_C(x;u,v)$$

In particular:

$$\overline{f}''_C(x;u,v) = \limsup_{y \to x} \overline{f}''_M(x;u,v).$$

Furthermore the following characterizations of $\overline{f}''_C$ and $\overline{f}''_M$ hold:

$$\overline{f}''_C(x;u,v) = \limsup_{y \to x, s,t \downarrow 0} \frac{\overline{\Delta}_2^{u,v} f(y;s,t)}{st}$$

where:

$$\overline{\Delta}_2^{u,v} f(y;s,t) = f(y+su+tv) - f(y+su) - f(y+tv) + f(y)$$

and:

$$f''_M(x;u,v) = \sup_{z_1,z_2 \in \mathbb{R}^n} \limsup_{s \downarrow 0} \frac{\overline{\Delta}_2^{u,v,z_1,z_2} f(x;s)}{s^3}$$

where:

$$\overline{\Delta}_2^{u,v,z_1,z_2} f(x;s,u,v,z_1,z_2) = f(x+su+sz_1+s^2v+s^2z_2) - f(x+su+sz_1+s^2z_2)$$
$$-f(x+sz_1+s^2v+s^2z_2) + f(x+sz_1+s^2z_2).$$

For a $C^{1,1}$ function on $\mathbb{R}^n$ the generalized Hessian, defined in [21] is given by:

$$\partial^2_H f(x_0) := co\{M : M = \lim \nabla^2 f(x_i) : x_i \to x_0, \nabla^2 f(x_i) \ exists\}.$$

Now suppose that $(u, v) \to \overline{f}''_H(x; u, v)$ is the support functional of the multifunction $x \to \partial^2_H f(x)$. It is easy to see (see [9]) that $\partial^2_M f(x)(u) \subseteq \partial^2_H f(x)u$ and $\overline{f}''_M(x; u, v) \leq \overline{f}''_H(x; u, v)$ and that $\partial^2_C f(x)(u) = \partial^2_H f(x)u$ and $\overline{f}''_C(x; u, v) = \overline{f}''_H(x; u, v)$. Hence we have $\partial^2_M f(x)(u) \subseteq \partial^2_H f(x)u = \partial^2_C f(x)(u)$ and $\overline{f}''_M(x; u, v) \leq \overline{f}''_H(x; u, v) = \overline{f}''_C(x; u, v)$.

Example 2.2. Let $g : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable on Ω and consider $f(x) = [g^+(x)]^2$ where $g^+(x) = \max\{g(x), 0\}$. Clearly f is $C^{1,1}$ on Ω and it is easy to check that, for all $x_0 \in \Omega$, the $\partial^2_H f(x_0)$ is given by the following expression:

$$\partial^2_H f(x_0) = \begin{cases} \{2g(x_0)\nabla^2 g(x_0) + 2\nabla g(x_0)\nabla g(x_0)^T\} & \text{if } g(x_0) > 0 \\ \{0\} & \text{if } g(x_0) = 0 \\ \{2\alpha\nabla g(x_0)\nabla g(x_0)^T : \alpha \in [0, 1]\} & \text{if } g(x_0) < 0 \end{cases}$$

The following result recalls a Taylor expansion for these types of generalized derivatives.

Theorem 2.3. *[55] Let $f : \mathbb{R}^n \to \mathbb{R}$ be $C^{1,1}$. Then there exists $\xi \in (x, y)$ such that:*

$$f(y) - f(x) - < \nabla f(x), y - x > \in \frac{1}{2} < \partial^2_M f(\xi)(y - x), y - x >$$

$$\subseteq \frac{1}{2} < \partial^2_C f(\xi)(y - x), y - x >$$

2.2 Peano and Riemann generalized derivatives

Peano [44], studying Taylor expansion formula for real functions, introduced a concept of a higher order derivative of a function f at a point x known thereafter as Peano derivative. The works of Oliver [42], Evans and Weil [13] are surveys of Peano derivative. Further properties of Peano derivatives are given in [17]. Investigating the convergence of trigonometric series, Riemann [46] introduced higher order derivatives based on divided differences. Riemann derivatives are further developed and modified in the works of other authors like De La Vallée-Poussin or Denjoy [11, 12]. They take a central place in the trigonometric series theory. In many works Peano and Riemann derivatives are compared. Some further aspects in this direction are presented by Guerraggio, Rocca [18] and Ginchev [16]. Recently comparison results have been published by Ash [1], Humke and Laczkovich [22] and others. The use of Peano derivative in $C^{1,1}$ optimization problems is due to Liu [34, 35, 36, 37]. We now recall the definitions and some properties which will be useful in the sequel.

Definition 2.1. The second Riemann derivative of f at a point $x \in \Omega$ in the direction $d \in \mathbb{R}^n$ is defined as:

$$f''_R(x;d) = \lim_{t \downarrow 0^+} \frac{\Delta_2^d f(x;t)}{t^2},$$

if this limit exists.

Similarly the upper and the lower Riemann derivatives are given by:

$$\overline{f}''_R(x;d) = \limsup_{t \downarrow 0^+} \frac{\Delta_2^d f(x;t)}{t^2}, \underline{f}''_R(x;d) = \liminf_{t \downarrow 0^+} \frac{\Delta_2^d f(x;t)}{t^2},$$

From the characterization of $\overline{f}''_C$ it is clear that $\overline{f}''_R(x;d) \leq \overline{f}''_C(x;d)$.

Definition 2.2. Let f be a differentiable function. If there exist a number L such that:

$$\lim_{t \downarrow 0^+} 2\frac{f(x+td) - f(x) - t < \nabla f(x), d >}{t^2} = L,$$

then f is said to admit a second Peano derivative at x in the direction d. The number L is said the second Peano derivative of f at x in the direction d and it will be denoted by $f''_P(x;d)$.

Similarly the upper and lower Peano derivatives are given by:

$$\overline{f}''_P(x;d) = \limsup_{t \downarrow 0^+} 2\frac{f(x+td) - f(x) - t < \nabla f(x), d >}{t^2},$$

and:

$$\underline{f}''_P(x;d) = \liminf_{t \downarrow 0^+} 2\frac{f(x+td) - f(x) - t < \nabla f(x), d >}{t^2}.$$

In [34] is proved that $\overline{f}''_P(x;d) \leq \overline{f}''_C(x;d)$. It is well known that the existence of the ordinary second directional derivative of f at x in the direction d, $f''(x;d)$ implies the existence of $f''_P(x;d)$ and this in turn implies the existence of $f''_R(x;d)$. However the existence of $f''_P(x;d)$ does not imply the existence of the second ordinary directional derivatives. In fact if we consider the function:

$$f(x) = \begin{cases} x^3 \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

then f has first order usual derivative in a neighborhood of $x = 0$ and a second order Peano derivative $f''_P(0) = 0$ but does not possess the second order usual derivative $f''(0)$.

Now let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be a function of class $C^{1,1}$. This hypothesis does not imply the existence of Peano and Riemann derivatives at every point of Ω but, from Rademacher's theorem, we can assure the existence for almost everywhere $x \in \Omega$. However the upper and lower Peano and Riemann derivatives are well defined and bounded $\forall x \in \Omega$.

3 Second order generalized derivatives and optimality conditions

The aim of this section is to establish some relations among generalized derivatives for $C^{1,1}$ functions and to show some optimality conditions for constrained and unconstrained optimization problems. The following result states two chains of inequalities among different definitions of generalized derivatives. Furthermore, the smallness of Peano derivative makes the corresponding optimality conditions sharper than those obtained by the other definitions.

Theorem 3.1. *Let f be a function of class $C^{1,1}$ at x_0. Then:*

i) $\overline{f}''_P(x_0;d) \leq \overline{f}''_D(x_0;d) \leq \overline{f}''_M(x_0;d,d) \leq \overline{f}''_C(x_0;d,d)$.

ii) $\overline{f}''_P(x_0;d) \leq \overline{f}''_R(x_0;d) \leq \overline{f}''_M(x_0;d,d) \leq \overline{f}''_C(x_0;d,d)$.

Proof. i) It is only necessary to prove the inequality $\overline{f}''_P(x_0;d) \leq \overline{f}''_D(x_0;d)$. If we take the function $\phi_1(t) = f(x_0+td) - t\nabla f(x_0)d$ and $\phi_2(t) = t^2$, applying Cauchy's theorem, we obtain:

$$2\frac{f(x_0+td) - f(x_0) - t < \nabla f(x_0), d >}{t^2} = 2\frac{\phi_1(t) - \phi_1(0)}{\phi_2(t) - \phi_2(0)} =$$

$$2\frac{\phi_1'(\xi)}{\phi_2'(\xi)} = \frac{\nabla f(x_0+\xi d)d - \nabla f(x_0)d}{\xi},$$

where $\xi = \xi(t) \in (0,t)$, and then[2] $\overline{f}''_P(x_0;d) \leq \overline{f}''_D(x_0;d)$.

ii) It is only necessary to prove the inequalities $\overline{f}''_P(x_0;d) \leq \overline{f}''_R(x_0;d) \leq \overline{f}''_M(x_0;d,d)$. Concerning the first inequality, from the definition of $\overline{f}''_P(x_0;d)$ we have:

$$f(x_0+td) = f(x_0) + t\nabla f(x_0)d + \frac{t^2}{2}\overline{f}''_P(x_0;d) + g(t)$$

where $\limsup_{t\to 0^+} \frac{g(t)}{t^2} = 0$ and:

$$f(x_0+2td) = f(x_0) + 2t\nabla f(x_0)d + 2t^2\overline{f}''_P(x_0;d) + g(2t)$$

where $\limsup_{t\to 0^+} \frac{g(2t)}{t^2} = 4\limsup_{t\to 0^+} \frac{g(2t)}{4t^2} = 0$. Then:

$$\frac{f(x_0+2td) - 2f(x_0+td) + f(x_0)}{t^2} = \frac{t^2\overline{f}''_P(x_0;d) + g(2t) - g(t)}{t^2} \geq$$

$$\overline{f}''_P(x_0;d) + \limsup_{t\to 0^+}\frac{g(2t)}{t^2} - \limsup_{t\to 0^+}\frac{g(t)}{t^2}.$$

[2]If $g(t) = h(\xi(t))$, $\xi(t) \downarrow 0^+$ when $t \downarrow 0^+$, then $\limsup_{t\downarrow 0^+} g(t) = \limsup_{t\downarrow 0^+} h(\xi(t)) \leq \limsup_{\xi\downarrow 0^+} h(\xi)$.

Then $\overline{f}''_R(x_0; d) \geq \overline{f}''_P(x_0; d)$. For the second inequality, we define $\phi_1(t) = f(x_0 + 2td) - 2f(x_0 + td)$ and $\phi_2(t) = t^2$. Then, by Cauchy's theorem, we obtain:

$$\frac{f(x_0 + 2td) - 2f(x_0 + td) + f(x_0)}{t^2} = \frac{\phi_1(t) - \phi_1(0)}{\phi_2(t) - \phi_2(0)} =$$

$$\frac{\phi_1'(\xi)}{\phi_2'(\xi)} = \frac{\nabla f(x_0 + 2\xi d)d - \nabla f(x_0 + t\xi)}{\xi},$$

where $\xi = \xi(t) \in (0, t)$, and then $\overline{f}''_R(x_0; d) \leq \overline{f}''_M(x_0; d, d)$.

□

Consider now the following *unconstrained* [3] *optimization problem*:

$$UP) \qquad \min_{x \in A} f(x)$$

where A is an open subset of $\mathbb{R}^n$.

Theorem 3.2. *[35] If $x_0 \in A$ is a local minimum point for problem UP) then $\nabla f(x_0) = 0$ and $\underline{f}''_P(x_0; d) \geq 0$, $\forall d \in S^1$.*

Theorem 3.3. *[35] Let $x_0 \in A$. If $\nabla f(x_0) = 0$ and $\underline{f}''_P(x_0; d) > 0$, $\forall d \in \mathbb{R}^n$, $d \neq 0$, then x_0 is a strict local minimum point for problem UP).*

Consider now the following inequality and equality constrained optimization problem:

$$CP) \qquad \min f(x)$$

$$\text{subject to x} \in S = \{x : h_k(x) = 0, k = 1 \ldots m, g_j(x) \leq 0, k = 1 \ldots l\}$$

where f, h_k, $k = 1 \ldots m$ and g_j, $j = 1 \ldots l$, are $C^{1,1}$ functions. Suppose that S is nonempty and let x_0 be a local minimum point for problem CP). Moreover, assume the following constraint qualification:

$$H) \qquad \nabla g_j(x_0), j \in J(x_0), \nabla h_k(x_0), k = 1 \ldots m, \text{are linearly independent},$$

where $J(x_0) = \{j : g_j(x_0) = 0\}$, is satisfied. Then there exists a vector $(\lambda_1, \ldots, \lambda_l, \mu_1, \ldots \mu_m) \in \mathbb{R}^{l+m}$ such that the Kuhn-Tucker optimality conditions:

$$1) \qquad \nabla f(x_0) + \sum_{j=1}^{l} \lambda_j \nabla g_j(x_0) + \sum_{k=1}^{m} \mu_k \nabla h_k(x_0) = 0,$$

$$2) \qquad \lambda_j \geq 0, \lambda_j g_j(x_0) = 0, j = 1 \ldots l,$$

are satisfied. To get the second order condition, we associate with each multiplier $\lambda = (\lambda_1, \ldots, \lambda_l)$, a set $G(\lambda)$ defined as follows:

$$G(\lambda) = \{x \in \mathbb{R}^n : g_j(x) = 0 \text{ when } \lambda_j > 0, g_j(x) \leq 0 \text{ when}$$

[3] The following optimality conditions are obtained by the notion of Peano's derivative and due to Liu[34, 35, 36, 37]. Further conditions can be found in [21, 55].

$$\lambda_j = 0, h_k(x) = 0, k = 1 \dots m\}$$

and denote the cone of feasible directions to $G(\lambda)$ at x_0 by:

$$F(G(\lambda), x_0) = \{d : \exists \delta > 0 \text{ s.t.} \forall \theta \in (0, \delta], x = x_0 + \theta d \in G(\lambda)\}.$$

If we express the usual Lagrangian function by:

$$L(x; \lambda) = f(x) + \sum_{j=1}^{l} \lambda_j g_j(x) + \sum_{k=1}^{m} \mu_k h_k(x)$$

where $\lambda = (\lambda_1, \dots, \lambda_l)$ and $\mu = (\mu_1, \dots \mu_k)$ and denote the lower generalized second order Peano's derivative of $L(\cdot, \lambda, \mu)$ at x_0 by $\underline{L}''_x(x_0, \lambda, \mu; d)$. The following result states a necessary optimality condition for problem CP).

Theorem 3.4. *[35] Let x_0 a local minimum point of CP) and let H) hold. Then for each Lagrangian multiplier vector (λ, μ) satisfying 1) and 2) at x_0, for each $d \in F(G(\lambda), x_0)$ we have $\underline{L}''_x(x_0, \lambda, \mu; d) \geq 0$.*

If we define the tangent cone to S at x_0 by:

$$T(S, x_0) = \{d : \exists t_i, t_i \downarrow 0^+, d_i \to d : x_0 + t_i d \in S, \forall i\}$$

then we have the second order sufficient condition for the problem CP).

Theorem 3.5. *[35] Let f, g_j, $j = 1 \dots l$, and h_k, $k = 1 \dots m$, be $C^{1,1}$ functions at $x_0 \in S$. If there exists a Kuhn-Tucker multiplier vector (λ, μ) satisfying 1) and 2) at x_0 and if for each $d \in T(S, x_0)$, $d \neq 0$, and $\underline{L}''_x(x_0, \lambda, \mu; d) > 0$, then x_0 is a strict local minimum point of problem CP).*

4 Numerical methods for $C^{1,1}$ optimization problems

The aim of this section is to show some numerical methods, based on a generalized Newton's method, for solving $C^{1,1}$ unconstrained optimization problems. So we consider the following optimization problem:

$$\min_{x \in \mathbb{R}^n} f(x)$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a function of class $C^{1,1}$. The generalized Newton's method for this problem is:

$$x_{k+1} = x_k - V_k^{-1} \nabla f(x_k)$$

where $V_k \in \partial^2_C f(x_k)$. We will use this procedure to approximate the solutions of the nonsmooth equation $\nabla f(x) = 0$ and we will recall convergence results under the semismoothness property. According to the above definition, $\nabla f : D \subset \mathbb{R}^n \to \mathbb{R}$ is said to be semismooth at x if ∇ is locally Lipschitzian at x and:

$$\lim_{V \in \partial^2_C f(x+th'), h' \to h, t \to 0} Vh'$$

exists for any $h \in \mathbb{R}^n$. Clearly if ∇f is semismooth at x, then ∇f is directionally differentiable at x ([53]) and for any $V \in \partial_C^2 f(x+h)$,

$$Vh - (\nabla f)'(x) = o(\|h\|).$$

Similarly, we have:

$$h^t V h - f''(x; h) = o(\|h\|^2).$$

The local convergence result of the previous procedure is the following:

Theorem 4.1. *[53] Suppose that f is of class $C^{1,1}$ and ∇f is semismooth at x^*, x_k is sufficiently closed to x^*, where x^* is a local minimizer of the optimization problem, $V \in \partial_C^2 f(x^*)$ is positive definite. Then the generalized Newton's iteration is well defined and converges to x^* with a superlinear rate.*

Now let us give the global convergence theorem of the generalized Newton's method with the exact line search. Consider the generalized Newton's iteration:

$$x_{k+1} = x_k - \alpha_k V_k^{-1} \nabla f(x_k)$$

where α_k is a steplenght factor from the exact line search.

Theorem 4.2. *[53] Suppose that f is a $C^{1,1}$ function on the level set*

$$L(x_0) = \{x \in \mathbb{R}^n : \|x - x_0\| \leq r\}$$

and ∇f is semismooth at x^*. Also suppose that $V \in \partial_C^2 f(x)$, V is positive definite, $\forall x \in L(x_0)$, and satisfies:

$$h^T V(x) h \geq m\|h\|^2, \forall x \in L(x_0), h \in \mathbb{R}^n$$

where the constant $m > 0$. Then the sequence x_k generated by the above generalized iteration with the exact line search satisfies:

- *either x_k is a finite sequence and $\nabla f(x_k) = 0$ for some k*
- *or x_k is an infinite sequence and $\nabla f(x_k) \to 0$, hence x_k converge to the unique minimizer x^* of f.*

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