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$C^{1,1}$ vector optimization problems and Riemann derivatives

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Abstract

In this paper we introduce a generalized second-order Riemann-type derivative for $C^{1,1}$ vector functions and use it to establish necessary and sufficient optimality conditions for vector optimization problems. We show that these conditions are stronger than those obtained by means of the second-order subdifferential in Clarke's sense, considered e.g. by Guerraggio and Luc ([4] and [5]).

Key words: Vector optimization, Riemann-type directional derivatives, Second-order optimality conditions.

Math. Subject Classifications: 90C29, 90C30, 49J52.

1 Introduction

The class of $C^{1,1}$ functions, that is differentiable scalar functions whose derivatives are locally Lipschitz was first brought into attention by Hiriart-Urruty, Strodiot and Hien Nguyen [6]. The need for investigating such functions, as pointed out in [6] and [7], comes from the fact that several problems of applied mathematics including variational inequalities, semiinfinite programming, iterated local minimization by decomposition etc., involve differentiable functions with no hope of being twice differentiable. By introducing generalized Hessian matrices with the help of Clarke's generalized Jacobians, the authors of [6] succeeded in extending Taylor's expansion and exploited it to derive second-order optimality conditions for scalar problems with data from this class of functions. Further applications were developed in [7], [11], [12], [15], [16] and [17].
The analysis has been generalized to vector functions by Guerraggio and Luc in [4], where by means of Clarke's second-order subdifferential second order necessary and sufficient optimality conditions for unconstrained vector optimization problems are established. In [5] the same authors also give second-order optimality conditions for constrained vector problems.

In this paper a generalized Riemann derivative for $C^{1,1}$ vector functions is introduced. By means of this derivative we give necessary and sufficient second-order optimality conditions for unconstrained vector optimization problems. It will be proved that these conditions are stronger than those given in [4]. When f is a scalar $C^{1,1}$ function the obtained optimality conditions reduce to those proved by Ginchev and Guerraggio [3].

2 Preliminary concepts

A function f from $\mathbf{R}^m$ to $\mathbf{R}^n$ is said of class $C^{0,1}$ at $x^0 \in \mathbf{R}^m$ when it is locally Lipschitz at x^0. We will say that f is of class $C^{0,1}$ when f is locally Lipschitz at any point of $\mathbf{R}^m$. If f is locally Lipschitz at x^0, then, according to Rademacher's theorem, it is almost everywhere differentiable in a neighborhood of x^0. Hence Clarke's generalized Jacobian of f at $x^0 \in \mathbf{R}^m$, denoted by $\partial f(x^0)$ is given by the set:

$$\partial f(x^0) = cl\ conv\{\lim f'(x_i) : x_i \to x^0,\ f'(x_i)\ \ exists\},$$

where f' denotes the Jacobian of f and $cl\ conv\{\dots\}$ stands for the closed convex hull of the set under the parentheses. Now assume that f is a differentiable vector functions from $\mathbf{R}^m$ to $\mathbf{R}^n$ whose derivative is of class $C^{0,1}$ at x^0. In this case we also say that f is of class $C^{1,1}$ at x^0. We say that f is of class $C^{1,1}$ when it is of class $C^{1,1}$ at any point of $\mathbf{R}^m$. Denote by f'' the Jacobian of the function $f' : \mathbf{R}^m \to \mathbf{R}^{m\times n}$. The Clarke's generalized Jacobian of f' at x^0 is then denoted by $\partial^2 f(x^0)$ and called the second-order subdifferential of f at x^0, more precisely

$$\partial^2 f(x^0) = cl\ conv\{\lim f''(x_i) : x_i \to x^0,\ \ f''(x_i)\ exists\}.$$

Thus $\partial^2 f(x^0)$ is a subset of the finite dimensional space $L(\mathbf{R}^m, L(\mathbf{R}^m, \mathbf{R}^n))$ of linear operators from $\mathbf{R}^m$ to the space $L(\mathbf{R}^m, \mathbf{R}^n)$ of linear operators from $\mathbf{R}^m$ to $\mathbf{R}^n$. The elements of $\partial^2 f(x^0)$ can therefore be viewed as bilinear functions on $\mathbf{R}^m \times \mathbf{R}^m$ with values in $\mathbf{R}^n$. For the case $n = 1$, the terminology "generalized Hessian matrix" was used in [6] to denote the set $\partial^2 f(x^0)$. By the previous construction the second-order subdifferential enjoys all properties of the generalized Jacobian. For instance $\partial^2 f(x_0)$ is a nonempty convex compact set of the space $L(\mathbf{R}^m, L(\mathbf{R}^m, \mathbf{R}^n))$ and the set-valued map $x \to \partial^2 f(x)$ is upper semicontinuous (u.s.c.). Let $u \in \mathbf{R}^m$; in the following we will denote by Lu the value of a linear operator $L : \mathbf{R}^m \to \mathbf{R}^n$ at the point $u \in \mathbf{R}^m$ and by $H(u, v)$ the value of a bilinear operator $H : \mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}^n$ at the point $(u, v) \in \mathbf{R}^m \times \mathbf{R}^m$. So we will set:

$$\partial f(x)(u) = \{Lu :\ L \in \partial f(x)\}$$

and

$$\partial^2 f(x)(u, u) = \{H(u, u) :\ H \in \partial^2 f(x)\}$$

We recollect some important properties from [4] and [5]:

(i) Mean value Theorem: Let f be of class $C^{0,1}$ and $a,\ b \in \mathbf{R}^m$. Then:

$$f(b) - f(a) \in cl\ conv\{\partial f(x)(b - a) : x \in [a, b]\},$$

where $[a, b] = conv\{a, b\}$;

(ii) TAYLOR'S EXPANSION: Let f be of class $C^{1,1}$ and $a,\ b \in \mathbf{R}^m$. Then:

$$f(b)-f(a) \in f'(a)(b-a)+\frac{1}{2}cl\ conv\{\partial^2 f(x)(b-a,b-a) : x \in [a,b]\}.$$

In [4] and [5] Guerraggio and Luc have given necessary and sufficient optimality conditions for vector optimization problems, expressed by means of $\partial^2 f(x)$.
In the following f will always denote a function of class $C^{1,1}$ at the considered point x^0.

Now we set:

$$\Delta_R^2 f(x^0,t,u)=\frac{f(x^0+2tu)-2f(x^0+tu)+f(x^0)}{t^2}.$$

The following theorem can be easily deduced from theorem 2.1 in [8] and characterizes functions of class $C^{1,1}$ in terms of $\Delta_R^2 f(x^0,t,u)$.

Theorem 2.1 *Assume that the function $f : \mathbf{R}^m \to \mathbf{R}^n$ is bounded on a neighborhood of the point $x^0 \in \mathbf{R}^m$. Then f is of class $C^{1,1}$ at x^0 if and only if there exist neighborhoods U of x^0 and V of $0 \in \mathbf{R}$ and a constant $M \geq 0$ such that $\|\Delta_R^2 f(x,t,u)\| \leq M$ for every $x \in U$, $t \in V\backslash\{0\}$ and $u \in S^1 = \{u \in \mathbf{R}^m : \|u\| = 1\}$.*

Definition 2.1 *The second upper Riemann derivative of the function f at the point $x^0 \in \mathbf{R}^m$ in the direction $u \in \mathbf{R}^m$ is defined as:*

$$f_R''(x^0,u)=Limsup_{t\to 0^+}\Delta_R^2 f(x^0,t,u),$$

where Limsup denotes the upper limit of sets in the sense of Kuratowski, that is the set of all cluster points of sequences $\Delta_R^2 f(x^0,t_k,u)$, taken as $t_k \to 0^+$.

Remark 2.1 B. Riemann introduced (for scalar functions) the homonymous notion of second-order derivative while he was studying the convergence of trigonometric series [14]. If g is a function from $\mathbf{R}$ to $\mathbf{R}$, the second-order Riemann derivative of g at the point $x \in \mathbf{R}$ is given by:

$$\lim_{t\to 0^+}\frac{g(x+2t)-2g(x+t)+g(x)}{t^2},$$

if this limit exists. Taking lim sup or lim inf instead of lim one obtains upper and lower Riemann derivatives. For properties and applications of Riemann derivatives one can see [1], [2], [13] and [18]. Hence the previous definition generalizes the notion of Riemann derivative to functions from $\mathbf{R}^m$ to $\mathbf{R}^n$.

The following theorem states basic properties of $f_R''(x^0,u)$.

Theorem 2.2 *$f_R''(x^0,u)$ is a nonempty and compact subset of $\mathbf{R}^n$.*

Proof. The thesis is an obvious consequence of theorem 2.1. □

The next result links the second upper Riemann derivative to $\partial^2 f(x)$.

Theorem 2.3 $f''_R(x^0, u) \subseteq \partial^2 f(x^0)(u, u)$.

Proof. Applying Taylor's expansion we can write for $t > 0$ "small enough":

$$f(x^0+2tu)-f(x^0+tu) \in tf'(x^0+tu)u+\frac{t^2}{2} cl\ conv\{\partial^2 f(x)(u, u) : x \in [x^0+tu, x^0+2tu]\}$$

and:

$$f(x^0) - f(x^0 + tu) \in -tf'(x^0 + tu)u + \frac{t^2}{2} cl\ conv\{\partial^2 f(x)(u, u) : x \in [x^0 + tu, x^0]\}.$$

Hence, by addition:

$$\Delta^2_R f(x^0, t, u) \in \frac{1}{2} cl\ conv\{\partial^2 f(x)(u, u) : x \in [x^0 + tu, x^0 + 2tu]\}+$$

$$\frac{1}{2} cl\ conv\{\partial^2 f(x)(u, u) : [x^0 + tu, x^0]\}.$$

Since the map $x \to \partial^2 f(x)$ is u.s.c., then for any $\varepsilon > 0$, there exists a neighborhood U of x^0 such that whenever $x \in U$ it holds:

$$\partial^2 f(x)(u, u) \subseteq \partial^2 f(x^0)(u, u) + \varepsilon B,$$

where B is the closed unit ball in $\mathbf{R}^n$. So, for t "small enough" we have:

(i) $cl\ conv\{\partial^2 f(x)(u, u) : x \in [x^0 + tu, x^0 + 2tu]\} \subseteq cl\ conv\,[\partial^2 f(x^0)(u, u) + \varepsilon B] = \partial^2 f(x^0)(u, u) + \varepsilon B$;

(ii) $cl\ conv\{\partial^2 f(x)(u, u) : x \in [x^0 + tu, x^0]\} \subseteq cl\ conv\,[\partial^2 f(x^0)(u, u) + \varepsilon B] = \partial^2 f(x^0)(u, u) + \varepsilon B$.

Hence we have, for t "small enough":

$$\Delta^2_R f(x^0, t, u) \in \partial^2 f(x^0)(u, u) + 2\varepsilon B.$$

If $t_k \to 0^+$ is a sequence such that $\Delta^2_R f(x^0, t_k, u) \to L \in \mathbf{R}^n$, then L is an element of $f''_R(x^0, u)$ and $L \in \partial^2 f(x^0)(u, u) + 2\varepsilon B$, since this set is compact. Since ε is arbitrary and $\partial^2 f(x^0)$ is closed we obtain:

$$L \in \partial^2 f(x^0)(u, u)$$

and the theorem in proved. $\square$

Remark 2.2 Since $\partial^2 f(x^0)(u, u)$ is convex, it follows also that $conv f''_R(x^0, u) \subseteq \partial^2 f(x^0)(u, u)$.

Remark 2.3 The forthcoming Example 3.1 shows that the inclusion in theorem 2.3 can be strict.

3 Necessary optimality conditions for weakly efficient and ideal solutions

In this section we prove second-order necessary optimality conditions for unconstrained vector optimization problems, which are stronger than those provided by Guerraggio and Luc [4].
Assume that the space $\mathbf{R}^n$ is partially ordered by a closed, convex, pointed cone C, with a nonempty interior and denote by A^c the complement of the set A.
Let M be any of the cones C^c, $C \backslash \{0\}$, and $int\ C$. The unconstrained vector optimization problem corresponding to the pair (f, M) is written as:

$$min_M f(x), \qquad x \in \mathbf{R}^m,$$

which amounts to finding a point $x^0 \in \mathbf{R}^m$ (called the optimal solution) such that there is no $x \in \mathbf{R}^m$ with $f(x) \in f(x^0) - M$. If this is true for all x in some neighborhood of x^0, then we call x^0 a local optimal solution. The optimal solutions of the vector problem corresponding to (f, C^c) (respectively $(f, C\backslash\{0\})$ and $(f, int\ C)$ are called ideal solutions (respectively, efficient solutions and weakly efficient solutions). It follows directly from the definition that x^0 is a local ideal solution if and only if there is a neighborhood $U \subset \mathbf{R}^m$ of x^0 such that:

$$f(x) - f(x^0) \in C, \quad \forall x \in U.$$

Guerraggio and Luc [4] have proved necessary and suffcicient optimality conditions for vector problems, that we summarize in the following theorems.

Theorem 3.1 *(i) Let $x^0 \in \mathbf{R}^m$ be a local weakly efficient solution. Then the following conditions hold:*
a) $f'(x^0)u \in (-int\ C)^c$, $\forall u \in \mathbf{R}^m$;
b) $\partial^2 f(x^0)(u,u) \cap (-int\ C)^c \neq \emptyset$, for $u \in \mathbf{R}^m$ with $f'(x^0)u \in -(C \backslash int\ C)$.

(ii) Let $x^0 \in \mathbf{R}^m$ be a local ideal solution . Then the following conditions hold:
a) $f'(x^0) = 0$;
b) $\partial^2 f(x^0)(u,u) \cap C \neq \emptyset$, $\forall u \in \mathbf{R}^m$.

Theorem 3.2 *(i) Assume that one of the following conditions holds at a point $x^0 \in \mathbf{R}^m$:*
a) $f'(x^0)u \in (-C)^c$, $\forall u \in \mathbf{R}^m$;
b) $\partial^2 f(x^0)(u,u) \subseteq int\ C$, for $u \in \mathbf{R}^m$ such that $f'(x^0)u = 0$.
Then x^0 is a local efficient solution.

(ii) Assume that the following conditions hold at a point $x^0 \in \mathbf{R}^m$:
a) $f'(x^0) = 0$;
b) $\partial^2 f(x^0)(u,u) \subseteq int\ C$, $\forall u \in \mathbf{R}^m \backslash \{0\}$.
Then x^0 is a local ideal solution.

Now we prove second-order necessary optimality conditions for unconstrained vector optimization problems, expressed by means of Riemann derivatives.

Theorem 3.3 *Let $x^0 \in \mathbf{R}^m$ be a local weakly efficient solution. Then the following conditions hold:*

(i) $f'(x^0)u \in (-int\ C)^c$, *for every* $u \in \mathbf{R}^m$*;*

(ii) $f''_R(x^0,u) \cap (-int\ C)^c \neq \emptyset$, *for* $u \in \mathbf{R}^m$, *with* $f'(x^0)u \in -(C \backslash int\ C)$.

Proof. Condition i) has been given in theorem 3.1 and so we prove only condition ii). We begin observing that for any $t > 0$, $u \in \mathbf{R}^m$ and $n = 1, 2, 3, \ldots$, the following identity holds (see also [3]):

$$*) \quad f(x^0 + tu) - f(x^0) = t\frac{f(x^0 + \frac{t}{2^n}u) - f(x^0)}{\frac{t}{2^n}} + \frac{t^2}{2}\sum_{i=1}^{n}\frac{1}{2^i}\Delta^2_R f(x^0, \frac{t}{2^i}, u).$$

Let $t > 0$ and $u \in \mathbf{R}^m$ be fixed. Observe that as $n \to +\infty$, we have:

$$\frac{f(x^0 + \frac{t}{2^n}u) - f(x^0)}{\frac{t}{2^n}} \to f'(x^0)u$$

and therefore $\sum_{i=1}^{+\infty}\frac{1}{2^i}\Delta^2_R f(x^0, \frac{t}{2^i}, u)$ converges. We claim that $\forall \gamma > 0$, $\exists \delta = \delta(\gamma) > 0$, such that, $\forall t \in (0, \delta)$ it holds:

$$\sum_{i=1}^{+\infty}\frac{1}{2^i}\Delta^2_R f(x^0, \frac{t}{2^i}, u) \in conv\ \{f''_R(x^0, u) + \gamma B\}.$$

Let us begin observing that $\forall \gamma > 0$, $\exists \delta = \delta(\gamma) > 0$, such that:

$$\Delta^2_R f(x^0, t, u) \in f''_R(x^0, u) + \gamma B,$$

for $t \in (0, \delta)$. In fact, if ab absurdo this inclusion does not hold, we can find a number $\overline{\gamma} > 0$ and a sequence $t_k \to 0^+$ such that:

$$\Delta^2_R f(x^0, t_k, u) \in \left(f''_R(x^0, u) + \overline{\gamma}B\right)^c, \quad \forall k.$$

Without loss of generality, we can assume that $\Delta^2_R f(x^0, t_k, u)$ converges to a limit L and:

$$L \in cl\left(f''_R(x^0, u) + \overline{\gamma}B\right)^c,$$

so that $L \notin f''_R(x^0, u)$, which is absurd. Now, fix $\gamma > 0$, and ab absurdo assume that for some $t \in (0, \delta)$ it holds:

$$\sum_{i=1}^{+\infty}\frac{1}{2^i}\Delta^2_R f(x^0, \frac{t}{2^i}, u) \notin conv\{f''_R(x^0, y) + \gamma B\}.$$

Since the set on the right hand side is convex and compact it follows the existence of a vector $p \in \mathbf{R}^n$, $p \neq 0$ and a number $\beta \in \mathbf{R}$ such that:

$$\langle p, \sum_{i=1}^{+\infty}\frac{1}{2^i}\Delta^2_R f(x^0, \frac{t}{2^i}, u)\rangle \ > \beta,$$

while $\langle p, a\rangle < \beta$, for any $a \in conv\{f''_R(x^0, y) + \gamma B\}$. We have:

$$\langle p, \sum_{i=1}^{+\infty} \frac{1}{2^i}\Delta^2_R f(x^0, \frac{t}{2^i}, u)\rangle = \lim_{n\to+\infty} \sum_{i=1}^{n} \frac{1}{2^i}\langle p, \Delta^2_R f(x^0, \frac{t}{2^i}, u)\rangle.$$

For $t \in (0, \delta)$, we obtain $\Delta^2_R f(x^0, \frac{t}{2^i}, u) \in f''_R(x_o, y) + \gamma B$ and hence:

$$\langle p, \Delta^2_R f(x^0, \frac{t}{2^i}, u)\rangle < \beta, \quad \forall i.$$

It follows that:

$$\sum_{i=1}^{+\infty} \frac{1}{2^i}\langle p, \Delta^2_R f(x^0, \frac{t}{2^i}, u)\rangle < \sum_{i=1}^{+\infty} \frac{1}{2^i}\beta = \beta,$$

and so we get a contradiction. Now, assume that $u \in \mathbf{R}^m$ is such that $f'(x^0)u \in -(C\backslash int\ C)$, and ab absurdo suppose that:

$$f''_R(x^0, u) \cap (-int\ C)^c = \emptyset,$$

that is $f''_R(x^0, u) \subseteq -int\ C$. Since $f''_R(x^0, u)$ is compact and C is open and convex, we obtain the existence of a number $\gamma > 0$ such that:

$$conv[f''_R(x^0, u) + \gamma B] \subseteq -int\ C.$$

Hence, for t "small enough", we obtain:

$$f(x^0 + tu) - f(x^0) = tf'(x^0)u + \frac{t^2}{2}\sum_{i=1}^{+\infty} \frac{1}{2^i}\Delta^2_R f(x^0, \frac{t}{2^i}, u)$$

$$\in -(C\backslash int\ C) - int\ C \subseteq -int\ C,$$

which is absurd. So the proof is complete.

□

Remark 3.1 Since $f''_R(x^0, u) \subseteq \partial^2 f(x^0)(u, u)$, the necessary conditions of the previous theorem are stronger than those proved by Guerraggio and Luc [4] in terms of $\partial^2 f(x^0)(u, u)$. The same remark holds for theorem 3.4. Example 3.3 shows that the conditions in Riemann derivarives can work in cases where the conditions from theorem 3.1 are not efficient.

Example 3.1 *Consider the function $f : \mathbf{R}^n \to \mathbf{R}^n$ defined for $x = (x_1, \dots, x_n) \in \mathbf{R}^n$ as $f(x) = (\int_0^{x_1} |z|\, dz\,, \dots, \int_0^{x_n} |z|\, dz\,)$ and let $C = \mathbf{R}^n_+$. It is easy to see that the point $x^0 = 0 \in \mathbf{R}^n$ is not a local weakly efficient solution. We have $f'(x)u = (|x_1|u_1, \dots, |x_n|u_n)$ for $u = (u_1, \dots, u_n) \in \mathbf{R}^n$ and in particular $f'(x^0) = 0 \in \mathbf{R}^{n\times n}$. For the second-order subdifferential we have $\partial^2 f(x^0)(u, u) = I\, u_1^2 \times \cdots \times I\, u_n^2$, where $I = [-1, 1] \subset \mathbf{R}$. The second-order necessary conditions from Theorem 3.1 (i) are satisfied, therefore we cannot conclude on this base that x^0 is not a local* *weakly efficient point. For the second-order Riemann derivative we have* $f_R''(x^0, u) = (\text{sign}(u_1)\, u_1^2, \ldots, \text{sign}(u_n)\, u_n^2)$*. In particular if all the coordinates of* u *are negative we have* $f_R''(x^0, u) = (-u_1^2, \ldots, -u_n^2) \in -int\ C$*. Therefore for such* u *the second-order necessary conditions from Theorem 3.3 are not satisfied and on this base we can conclude that* x^0 *is not a local efficient point.*

The following theorem states necessary conditions for local ideal solutions.

Theorem 3.4 *Let* $x^0 \in \mathbf{R}^m$ *be a local ideal solution. Then the following conditions hold:*

(i) $f'(x^0) = 0$*;*

(ii) for every $u \in \mathbf{R}^m$ *we have* $conv\, f_R''(x^0, u) \cap C \neq \emptyset$*.*

Proof. Condition i) is stated in theorem 3.1 and so we prove only condition ii). Ab absurdo, assume that x^0 is a local ideal solution, but ii) does not hold, so that there exists a vector $u \in \mathbf{R}^m$ such that:

$$conv\, f_R''(x^0, u) \cap C = \emptyset,$$

that is $conv\, f_R''(x^0, u) \subseteq C^c$. Since $conv\, f_R''(x^0, u)$ is compact and C^c is open, there exists a number $\gamma > 0$ such that:

$$conv[f_R''(x^0, u) + \gamma B] = conv\, f_R''(x^0, u) + \gamma B \subseteq C^c.$$

From the proof of the previous theorem, we know that $\forall \gamma > 0$, $\exists \delta > 0$ such that:

$$\sum_{i=1}^{+\infty} \frac{1}{2^i} \Delta_R^2 f(x^0, \frac{t}{2^i}, u) \in conv\{f_R''(x^0, u) + \gamma B\},$$

$\forall t \in (0, \delta)$, and using identity *), we find that for t "small enough", $f(x^0 + tu) - f(x^0) \in C^c$, that is a contradiction. □

It easy to see that when a function $f : \mathbf{R}^m \to \mathbf{R}$ is considered, then from the previous theorems we recover the necessary conditions for a local extremum, proved by Ginchev and Guerraggio [3], as stated in the following result.

Corollary 3.1 *Let* $f : \mathbf{R}^m \to \mathbf{R}$ *be a function of class* $C^{1,1}$ *at a point* $x^0 \in \mathbf{R}^m$*. If* x^0 *is a local minimizer of the function* f*, then the following conditions hold:*

(i) $f'(x^0) = 0$*;*

(ii) $\limsup_{t \to 0^+} \Delta_R^2(x^0, t, u) \geq 0, \quad \forall u \in \mathbf{R}^m.$

4 Sufficient optimality conditions for efficient and ideal solutions

Before giving sufficient optimality conditions, we prove the following lemma.

Lemma 4.1

$$f''_R(x^0,u) = f''_r(x^0,u),$$

where $f''_r(x^0,u) = Limsup_{t\to 0^+,\ u'\to u}\Delta^2_R f(x^0,t,u')$*, that is the set of all cluster points of sequences* $\Delta^2_R f(x^0,t_k,u_k)$*, taken as* $t_k \to 0^+$ *and* $u_k \to u$*.*

Proof. The inclusion $f''_R(x^0,u) \subseteq f''_r(x^0,u)$ is obvious, so that we have only to prove the reverse inclusion. Let $L \in f''_r(x^0,u)$; hence there exist sequences $t_k \to 0^+$ and $u_k \to u$ as $k \to +\infty$, such that:

$$L = \lim_{k\to+\infty} \frac{1}{t_k^2}\Big(f(x^0+2t_k u_k) - 2f(x^0+t_k u_k) + f(x^0)\Big).$$

We have:

$$\frac{1}{t_k^2}\Big(f(x^0+2t_k u_k) - 2f(x^0+t_k u_k) + f(x^0)\Big)$$

$$= \frac{1}{t_k^2}\,[f(x^0+2t_k u_k) - 2f(x^0+t_k u_k) + f(x^0)$$

$$-f(x^0+2t_k u) + f(x^0+2t_k u) - 2f(x^0+t_k u) + 2f(x^0+t_k u)].$$

Without loss of generality we can assume that:

$$\frac{1}{t_k^2}\Big(f(x^0+2t_k u) - 2f(x^0+t_k u) + f(x^0)\Big) \to L'.$$

Applying the mean value theorem we have:

$$[f(x^0+2t_k u_k) - f(x^0+2t_k u) - 2(f(x^0+t_k u_k) - f(x^0+t_k u))]$$

$$= 2t_k\left[f'(x^0+2t_k u + 2\theta_k t_k(u_k-u))(u_k-u) - f'(x^0+t_k u+\theta'_k t_k(u_k-u))(u_k-u)\right],$$

where $\theta_k,\ \theta'_k \in (0,1)$. Since $f \in C^{1,1}$ we obtain:

$$\|2t_k\left[f'(x^0+2t_k u + 2\theta_k t_k(u_k-u))(u_k-u) - f'(x^0+t_k u+\theta'_k t_k(u_k-u))(u_k-u)\right]\|$$

$$\le 2Kt_k\|t_k u + 2\theta_k t_k(u_k-u) - \theta'_k t_k(u_k-u)\|\|u_k-u\|$$

$$\le 2Kt_k^2\|u\|\|u_k-u\| + 2Kt_k^2|2\theta_k-\theta'_k|\|u_k-u\|^2,$$

where K is a Lipschitz constant for f'. Hence it is easily seen that:

$$\frac{1}{t_k^2}\Big[f(x^0+2t_k u_k) - f(x^0+2t_k u) - 2(f(x^0+t_k u_k) - f(x^0+t_k u)\Big] \to 0.$$

It follows that $L = L'$ and the lemma is proved. □

Theorem 4.1 *Let f be a function of class $C^{1,1}$ and assume that at the point $x^0 \in \mathbf{R}^m$ for every $u \in S^1$ one of the following conditions holds:*

(i) $f'(x^0)u \in (-C)^c$*;*

(ii) $f''_R(x^0, u) \subseteq int\ C$*, for* $u \in S^1$ *such that* $f'(x^0)u \in -(C \backslash int\ C)$*.*

Then x^0 is a local efficient solution.

Proof. Assume, ab absurdo, that i) and ii) hold, but x^0 is not a local efficient solution. Then there exists a sequence $x_j \to x^0$ such that:

$$f(x_j) - f(x^0) \in -C \backslash \{0\}.$$

We can always write $x_j = x^0 + t_j u_j$, where $t_j \to 0^+$, $u_j \in S^1$ and without loss of generality one can think that $u_j \to u \in S^1$. For this u two possibilities are given:

(i) $f'(x^0)u \in (-C)^c$;

(ii) $f'(x^0)u \in -(C \backslash int\ C)$.

Assume that the first case holds. Then, since f is of class $C^{1,1}$ we have:

$$f'(x^0)u = \lim_{t \to 0^+} \frac{f(x^0 + tu) - f(x^0)}{t} = \lim_{j \to +\infty} \frac{f(x^0 + t_j u_j) - f(x^0)}{t_j}.$$

Since $(-C)^c$ is open, we obtain that $f(x^0 + t_j u_j) - f(x^0) \in (-C)^c$, for j large enough, which is a contradiction.
Assume now that the second case holds. As in the proof of theorem 3.3, using the previous lemma, we obtain that $\forall \gamma > 0$, there exists a number $\delta = \delta(\gamma) > 0$, such that $\forall t \in (0, \delta)$ and $u' \in U_\delta(u)$ it holds:

$$\sum_{i=1}^{+\infty} \frac{1}{2^i} \Delta_R^2 f(x^0, \frac{t}{2^i}, u') \in conv\ \{f''_R(x^0, u) + \gamma B\},$$

where $U_\delta(u)$ denotes a neighborhood of u of radius δ. Since $f''_R(x^0, u) \subseteq int\ C$, we obtain, for γ "small enough":

$$conv\{f''_R(x^0, u) + \gamma B\} \subseteq int\ C.$$

Now, for every j we can write:

$$f(x^0 + t_j u_j) - f(x^0) = t_j f'(x^0) u_j + \frac{t^2}{2} \sum_{i=1}^{+\infty} \frac{1}{2^i} \Delta_R^2 f(x^0, \frac{t_j}{2^i}, u_j)$$

and observe that for j "large enough" the left hand side in this equality is contained in:

$$(-C)^c \cup \{-(C \backslash int\ C)\} + int\ C \subseteq (-C)^c$$

and this is a contradiction. □

Theorem 4.2 *Let f be a function of class $C^{1,1}$ and assume that the following conditions hold at a point $x^0 \in \mathbf{R}^m$:*

(i) $f'(x^0) = 0$;

(ii) $f''_R(x^0, u) \subseteq int\ C$, *for every* $u \in S^1$. *Then x^0 is a local ideal solution.*

Proof. Ab absurdo, assume that x^0 is not a local ideal solution. Then there exists a sequence $x_j = x^0 + t_j u_j$, $u_j \in S^1$, $u_j \to u \in S^1$ such that $f(x_j) - f(x^0) \in C^c$. We have:

$$f(x_j) - f(x^0) = \sum_{i=1}^{+\infty} \frac{1}{2^i} \Delta_R^2 f(x^0, \frac{t_j}{2^i}, u_j).$$

By analogy to the previous theorem we can conclude that for j "large enough",

$$\sum_{i=1}^{+\infty} \frac{1}{2^i} \Delta_R^2 f(x^0, \frac{t_j}{2^i}, u_j) \in int\ C,$$

and this is a contradiction. □

Remark 4.1 The sufficient optimality conditions proved in the previous theorems are stronger than those provided by Guerraggio and Luc, since $f''_R(x^0, u) \subseteq \partial^2 f(x^0)(u, u)$.

Example 4.1 *Let $C = \mathbf{R}^2_+$ and consider the twice differentiable function $f : \mathbf{R} \to \mathbf{R}^2$ defined as:*

$$f(x) = (x^2, \int_0^x z^2 \sin \frac{1}{z} dz + cx^2),$$

where $c \in (0, 1/2)$. Then, at the local ideal point $x^0 = 0$ we have $f'(0) = 0$ and:

$$\partial^2 f(0)(u, u) = [(2u^2, (-1 + 2c)u^2), (2u^2, (1 + 2c)u^2)],$$

whenever $u \in \mathbf{R}$. Hence the sufficient condition on $\partial^2 f(0)(u, u)$ for 0 to be a local ideal solution is not satisfied. On the contrary we have :

$$f''_R(0, u) = (2u^2, 2cu^2), \quad \forall u \in \mathbf{R}$$

and so the sufficient condition on $f''_R(0, u)$ for f to be a local ideal solution is satisfied.

When f is a function from $\mathbf{R}^m$ to $\mathbf{R}$, the previous theorems provide the following sufficient optimality conditions for f (see also [3]).

Corollary 4.1 *Let $f : \mathbf{R}^m \to \mathbf{R}$ be a function of class $C^{1,1}$ at $x^0 \in \mathbf{R}^m$ and assume that at x^0 the following conditions hold:*

(i) $f'(x^0) = 0$;

(ii) $\liminf_{t \to 0^+} \Delta_R^2(x^0, t, u) > 0$, *for every* $u \in S^1$.

Then x^0 is a local minimizer of f.

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