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Stability of critical points for vector valued functions and Pareto efficiency

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Abstract

In this work we consider the critical points of a vector-valued function f , defined as in [17] and [23]. We study their stability in order to obtain a necessary condition for Pareto efficiency. We point out, by an example, that the classical notions of stability (concerning a single point) are not suitable in this setting. We use a stability notion for sets to prove that the counterimage of a minimal point for f is stable. This result is based on the study of a dynamical system defined by a differential inclusion. In the vector case this inclusion plays the same role as gradient system in the scalar setting.

1 Introduction

The present work deals with a dynamic approach to vector optimization, which extends to the vector case some techniques typical of scalar functions as the study of gradient systems (see [7]). In particular, we give a necessary condition, for a critical point of a vector-valued function, to be a Pareto efficient point. Indeed we prove that the counterimage of a minimal point, which is a subset of the set of efficient points, is stable in a suitable sense. We underline that we use a notion of stability involving a whole set instead of a single point (as in [17] and [23]). We also provide an example that shows how stability of a point fails to be a necessary condition even for a stronger notion than the classical Pareto efficiency. In order to obtain this result we adapt to our setting the approach outlined in [10], where a vector-valued Liapunov function is introduced for discrete set-valued dynamic systems.

There are some works known in the literature concerning critical points for vector-valued functions and their relations with vector optimization. S. Smale was the first to approach the vector optimization problem from a dynamical point of view (see [17], [23], [24], [22] and [21]). He also applied this approach to the economic theory (see [18], [19], [20] and [25]). This contributes are resumed in [26]. The same approach was adopted in various other works appeared in the seventies and in the early eighties ([16], [27], [13], [14], [2], [12]). This kind of study has recently received new attention: see works [5] and [6] which focus on algorithmical aspects and the more theoretical works [4] and [9] which develop a critical point theory for vector valued functions. We mention also [15] where the notion of pseudogradient is defined in the vector case. Finally we recall also [1] where differential equations are used to solve a vector optimization problem.

Now we recall some notations concerning the solutions of a vector optimization problem. Let Q be a nonempty subset of $\mathbb{R}^m$, we define the set

$$\text{Min}(Q) := \{y \in Q : (y - Q) \cap (-\mathbb{R}_+^m) = \{0\}\}$$

where $\mathbb{R}_+^m = \{y = (y_1, \dots, y_m) \in \mathbb{R}^m : x_i \geq 0 \ \forall i = 1, \dots, m\}$. The elements of $\text{Min}(Q)$ are called *Pareto minimal points* of Q . Now let $f = (f_1, \dots, f_m) : W \rightarrow \mathbb{R}^m$ be a function defined on a set W , we defined the set of *Pareto efficient points* of f by

$$\text{Eff}(f, W) := \{x \in W : f(x) \in \text{Min}(Q)\}.$$

We introduce also the set of *ideal Pareto minimal point*

$$\text{IMin}(Q) := \{y \in Q : Q \subset y + \mathbb{R}_+^m\}$$

and the correspondent set of *ideal Pareto efficient points*

$$\text{IEff}(f, W) := \{x \in W : f(x) \in \text{IMin}(Q)\}.$$

We refer to [8] for an exposition of the theory of vector optimization in a more general setting.

In this work we use extensively the notion of manifolds and some related topics (tangent space, differential map,...); for a general introduction to this subject see, e.g., [3].

2 Dynamic approach to vector optimization

In this section we recall the main features of the approach to vector optimization problem developed by S. Smale in the series of papers quoted above. Here we refer especially to [17] and [23].

Let W be an open subset of the euclidean space $\mathbb{R}^n$ or is, more generally, a C^1 -manifold ($\dim W < \infty$). Let $f = (f_1, \dots, f_m) : W \rightarrow \mathbb{R}^m$ be a function of class C^1 on W . Obviously $f_i : W \rightarrow \mathbb{R}$ denotes the i -th component of the function f . We study the following vector optimization problem

$$\min_{x \in W} f(x)$$

i.e. the problem concerning the search of the points of the set $\text{Eff}(f, W)$. Instead of dealing with this set directly, we define a set K_f easier to handle, that contains $\text{Eff}(f, W)$. The definition of this set is based on differential conditions, since it plays the role of the set of critical points for a real valued function. Following [23], we call the set K_f *critical Pareto set*. In order to state the definition of K_f we must introduce some notions.

Let $x \in W$, we define the set

$$C_x := \{v \in T_x(W) : Df(x)v \in -\text{int}\mathbb{R}_+^m\}.$$

It easy to see that C_x is a cone (without the vertex) included in the tangent space $T_x(W)$. Since $C_x = Df(x)^{-1}(-\text{int}\mathbb{R}_+^m)$, we have that C_x is an open convex cone. Naturally we can define a set-valued map $C : W \rightrightarrows \cup_{x \in W} T_x(W) = T(W)$.

Definition 2.1 *The Pareto critical set for the function f is defined by*

$$K_f := \{x \in W : C_x = \emptyset\}.$$

It easy to see that $K_f = \{x \in W : Du(x)[T_x(W)] \cap -\text{int}(\mathbb{R}_+^m) = \emptyset\}$.

We recall also the notion of admissible curve that has a central role in the definitions of stability.

Definition 2.2 *We say that a curve $x : (a, b) \rightarrow W$ is admissible (with respect to the function f) when it satisfies the following inclusion*

$$x'(t) \in C_{x(t)} \quad \text{for every } t.$$

Remark 2.3 *A curve $x : (a, b) \rightarrow W$ is admissible if and only if*

$$\frac{d}{dt} f_i(x(t)) < 0 \quad \forall t \in (a, b), \text{ for every } i = 1, \dots, m.$$

or equivalently

$$Df(x(t))x'(t) \in -\text{int}\mathbb{R}_+^m \text{ for every } t \in (a, b)$$

where $x'(t) = \frac{d}{dt}x(t) \in T_{x(t)}(W)$ is the tangent vector to the curve x at $x(t)$.

The following proposition shows that the set $\text{Eff}(f, W)$ is a subset of the Pareto critical set.

Proposition 2.4 *If $x \in \mathbb{R}^n$ is a Pareto minimal point for f , then $x \in K_f$.*

Proof. Let x be a Pareto minimal point.

Since

$$[Df(x)]v = \lim_{t \rightarrow 0} \frac{1}{t} [f(x + tv) - f(x)] \quad \forall v \in \mathbb{R}^n \quad (t \in \mathbb{R})$$

we immediately have that $[f'(x)]v \notin -\text{int}(\mathbb{R}_+^m)$ for every $v \in T_x(W)$, i.e. $x \in K_f$. ■

For a detailed exposition of this approach and related items see [17], [23] and [22]. We recall also that the work [21] deals with Pareto optimization problems with constraints.

3 Stability of critical points

This section is devoted to the study of stability of critical points. In [17] and in [23] two slightly different definitions of stable critical point are formulated. The aim of these definitions concerns the construction of a smaller set than K_f that contains the set of the efficient points of f .

Definition 3.1 ([17]) *A point $x \in K_f$ is stable (in the sense 1) if given a neighborhood U of x in K_f , there exists a neighborhood V of x in W such that if $x : [a, b) \rightarrow W$ is any admissible curve starting in V (i.e. $x(a) \in V$) and such that $\lim_{t \rightarrow b} x(t) = \tilde{x} \in K_f$, then $\tilde{x} \in U$.*

Definition 3.2 ([23]) *A point $x \in K_f$ is stable (in the sense 2) if given a neighborhood U of x in W , there exists a neighborhood V of x in U such that for any admissible curve $x : [a, b) \rightarrow W$, with $x(a) \in V$, we have $x([a, b)) \subset U$.*

Initially Smale guess that every Pareto efficient point (in particular of type 2) belong to the set of stable points (in the sense 2) (see [23]). Then he states an example that shows how this statement is not true (see [19], sec.3). In this paper we give a different example where it can be seen that there are efficient points which are not stable (both in sense 1 and 2). We point out that our example shows that also a Pareto ideal point is not stable.

Example 3.3 Let $W = \mathbb{R}^2$. We consider the function f defined in the following way

$$\begin{aligned} f : \quad \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x^1, x^2) &\longmapsto ((x^1)^2, (x^1)^2) . \end{aligned}$$

Then

$$Df((x^1, x^2)) = \begin{bmatrix} 2x^1 & 0 \\ 2x^1 & 0 \end{bmatrix} .$$

Moreover it easy to see that

$$f(\mathbb{R}^2) = \{(y^1, y^2) \in \mathbb{R}^2 : y^1 = y^2, y \geq 0\}$$

and

$$\begin{aligned} \{(0, 0)\} &= \text{Min}(f(\mathbb{R}^2)) = \text{IMin}(u(\mathbb{R}^2)) \\ \{(x^1, x^2) \in \mathbb{R}^2 : x^1 = 0\} &= \text{Eff}(f, \mathbb{R}^2) = \text{IEff}(f, \mathbb{R}^2); \end{aligned}$$

Finally we observe that

$$\{(x^1, x^2) \in \mathbb{R}^2 : x^1 = 0\} = K_f .$$

Now we prove that every point of K_f is not stable both in the sense 1 and 2. We can consider only the point $(0, 0)$ without loss of generality. Let $r > 0$ and $U = B_r((0, 0))$. We define the curve $x : [0, 1) \rightarrow \mathbb{R}^2$ by

$$x(t) = (\varepsilon(1 - t), -(r + 1)t) .$$

where ε is a positive real number. Since

$$x'_1(t) = (-\varepsilon, -(r + 1))$$

we have

$$\begin{aligned} Du(x_1(t))x'_1(t) &= \begin{bmatrix} 2\varepsilon(1 - t) & 0 \\ 2\varepsilon(1 - t) & 0 \end{bmatrix} \begin{bmatrix} -\varepsilon \\ -(r + 1) \end{bmatrix} \\ &= \begin{bmatrix} -2\varepsilon^2(1 - t) \\ -2\varepsilon^2(1 - t) \end{bmatrix} \in -\text{int}\mathbb{R}_+^2 \quad \forall t \in [0, 1) . \end{aligned}$$

therefore we conclude that x is an admissible curve starting from $x_1(0) = (\varepsilon, 0)$ and ending in $(0, -(r + 1)) \in K_f \setminus U$. Now it easy to see that, choosing a suitable ε , for every neighborhood V of $(0, 0)$ we have that x starts inside V but it does not both stay in U or end in U .

4 Stable set and Pareto efficient points

In this section we consider the notion of stable set instead of stability of single point and we prove that the counterimage of a minimal point, which is a subset of the set of efficient points, are stable. Here we follow some ideas introduced in [10]. This paper deals with discrete dynamic systems defined by a set-valued map: it studies the stable set with respect to this type of systems introducing a vector valued Liapunov function. In particular we adapt to our situation the Theorem 3.8 in [10]. In order to state our results we must recall some notions.

Let X be a linear metric space and $F : X \rightrightarrows X$ be a set-valued map. We consider the dynamic system defined by the following differential inclusion

$$x'(t) \in F(x(t)) \quad (1)$$

where $x : [a, b) \rightarrow X$.

Definition 4.1 *A nonempty subset Q of X is stable (with respect to (1)) if for every neighborhood U of Q there exists a neighborhood V of Q such that if, given x satisfying (1) with $x(a) \in V$, then $x([a, b)) \subset U$.*

It is easy to see that a point x is stable in the sense of Definition 3.2 if and only if the set $\{x\}$ is stable.

We also point out that in this section we consider a more general problem: the feasible region of the vector optimization problem is an arbitrary subset S of the manifold W , which is not necessarily finite dimensional and is modeled by a normed space E . In order to deal with this problem we have to adapt some notions introduced in Section 2.

We begin to recall the extension of the notion of tangent cone to the manifolds setting.

Definition 4.2 ([11]) *Let S be a nonempty subset of W . A vector $v \in T_x(W)$ is said to be tangent to S at $x \in S$ if there exists a chart $\varphi : U \rightarrow E$ of W with $x \in U$ such that*

$$\lim_{t \downarrow 0} \frac{d(\varphi(x) + tD\varphi(x)v, \varphi(U \cap S))}{t} = 0$$

where $d(\varphi(x) + tD\varphi(x)v, \varphi(U \cap S)) = \inf_{e \in \varphi(U \cap S)} \|\varphi(x) + tD\varphi(x)v - e\|_E$ and $D\varphi : T_x(W) \rightarrow E$ is the differential of the chart φ at $x \in U$. The set of all tangent vectors to S at $x \in S$ is denoted by $T_x S$.

The set $T_x S$ is a closed cone in the tangent space $T_x(W)$ (see [11]). Moreover $T_x S$ coincides to the whole tangent space $T_x(W)$ whenever $x \in \text{int} S$ and it reduces to the classical tangent cone when W is a normed vector space.

Let $f = (f_1, \dots, f_m) : W \rightarrow \mathbb{R}^m$ be a function of class C^1 on W . We introduce the set-valued map $\hat{C} : S \rightrightarrows T(W)$ defined by:

$$\hat{C}(x) := \{v \in T_x S : Df(x)v \in -\text{int}\mathbb{R}_+^m\} \cup \{0\}$$

and we consider the dynamic system

$$x'(t) \in \hat{C}(x(t)). \quad (2)$$

In the sequel we suppose that the set S satisfies some regularity assumptions that guarantee the continuity of the set-valued map $\hat{C}$, hence the existence of solutions of the Cauchy problem associated to 2. Although this problem has great relevance we do not deal with it in the present brief paper and we refer to [12]. For existence theorems for differential inclusions we also refer to [2].

Remark 4.3 *We can immediately observe that f decreases along the trajectories determined by the differential inclusion (2), i.e.*

$$\text{if } t_1 > t_2 \quad \text{then} \quad f_i(x(t_1)) \geq f_i(x(t_2)) \quad \text{for every } i = 1, \dots, m.$$

In this framework the Pareto critical set is defined by

$$\hat{K}_f := \{x \in S : \hat{C}_x = \{0\}\}.$$

Easily we can observe that the Proposition 2.4 remains true and that $K_f \cap S \subset \hat{K}_f$.

We observe that the differential inclusion (2) plays a role similar to gradient system in the scalar case (see, e.g., [7]).

Now we can state the main result of our note.

Theorem 4.4 *Let $S \subset W$ be a compact set. If $p = (p_1, \dots, p_m) \in \text{Min}(f(S))$ then the set*

$$f^{-1}(p) = \{x \in S : f(x) = p\}$$

is stable (with respect to (2)).

Proof. Let O be an open (with respect to the topology induced by the topology of W) subset of S such that $f^{-1}(p) \subset O$. It follows immediately

that $H = S - O$ is compact. Let $h \in H$, then there exist some integers k , $1 \leq k \leq m$ and a positive integer r such that

$$f_k(h) > p_k + \frac{1}{r}.$$

Therefore the family $\{U_{k,r} : 1 \leq k \leq m, r \in \mathbb{N} \setminus \{0\}\}$ where

$$U_{k,r} = \left\{ x \in S : f_k(x) > p_k + \frac{1}{r} \right\}$$

is an open cover of H . Then there exists a finite subcover $\{U_{k_j,r_j} : j \in \mathbb{N}, 1 \leq j \leq q\}$ that is a cover of H . Now we consider the set

$$V_{k_j,r_j} = \left\{ x \in S : f_{k_j}(x) < p_{k_j} + \frac{1}{r_j} \right\}$$

that is a neighborhood of $f^{-1}(p)$ for every $j = 1, \dots, q$. Let $V = \bigcap_{j=1}^q V_{k_j,r_j}$; V is a neighborhood of $f^{-1}(p)$ such that $V \subset O$. If we consider a trajectory $x : [a, b] \rightarrow S$ defined by (2) starting from $x_0 = x(a) \in V$ we have

$$f_{k_j}(x(t)) < p_{k_j} + \frac{1}{r_j} \quad \text{for every } j = 1, \dots, q;$$

hence $x([a, b]) \subset V \subset U$. ■

Remark 4.5 As observed in [10], we can drop the compactness assumption on S if we suppose that

$$V_{k_0,r_0} = \left\{ x \in S : f_{k_0}(x) < p_{k_0} + \frac{1}{r_0} \right\}$$

is compact for some k_0, r_0 .

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