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2003/2

UNIVERSITÀ DELL'INSUBRIA
FACOLTÀ DI ECONOMIA

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Printed in Italy in January 2003
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Via Ravasi 2, 21100 Varese, Italy

Convergence of the Minimal Sets under Convexity in Vector Optimization

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Abstract

We study the behaviour of the minimal sets of a sequence of convex sets $\{A_n\}$ converging to a given set A. Under suitable assumptions involving only the structure of the single sets A_n,we obtain the lower convergence of MinA_n to Min$A$. In a reflexive Banach space ordered by a closed convex cone with a weakly compact base, we consider a sequence of convex sets A_n Mosco-converging to a set A. In the more general setting of a normed linear space ordered by a closed convex based cone (without any assumptions on the compactness of the base), we consider the stronger notion of Attouch-Wets convergence of the sequence of convex sets $\{A_n\}$. We compare our theorems with existing results related to the same topic.

Key Words: Stability, vector optimization, set-convergences, convexity.

1 Introduction

Stability is a widely studied topic in the theory of vector optimization. In this field the study of the behaviour of minimal sets under perturbations plays an important role. The classical way to deal with stability in vector optimization is the study of the continuity properties of the marginal multifunction (see, e.g., [16], [13] and, among recent works [3]. The approach of our work is based on the notion of set-convergence: we consider a sequence of perturbed sets $\{A_n\}$ converging to a given set A and we study the convergence of the minimal sets MinA_n. The same problem is discussed in [2] and in [11] (see also the references therein).

The originality of our work lies in the use of convexity assumptions to establish stability results in the sense of convergence of minimal sets. Moreover we avoid any requirement on the sequence of perturbed sets $\{A_n\}$ as a whole. In [2] and [11] the results on the convergence of minimal sets are obtained using some "compactness" or "boundedness" assumption involving the whole sequence $\{A_n\}$. We substitute these hypotheses with the requirement that every single set A_n is convex. Our assumptions seems to be easier to test. In any case, in the last section, we furnish some examples to prove that our assumptions do not imply those introduced in the above quoted papers. The fact that convexity plays an important role in stability theory in vector optimization is not surprising: also in the scalar case convexity is deeply related to stability.

The setting of the first result in Section 3 is a reflexive Banach space ordered by a closed and convex cone with a weakly compact base. We consider a family of convex perturbations $\{A_n\}$ converging in the sense of Mosco to a given set A. It is known (see [7]) that the assumptions on the order structure are restrictive since they are not satisfied by the nonnegative orthants of the most common infinite dimensional normed vector lattices (e.g., l^p and L^p, $1 \leq p \leq \infty$). Moreover we obtain a Kuratowski convergence result on MinA_n only in the weak topology. In fact this theorem allows us to obtain a finite dimensional corollary which is original with respect to the existing literature.

In order to avoid the restrictive environement of the previous section and to strengthen the convergence in the thesis we require a stronger convergence of the sequence $\{A_n\}$to A, the so called Attouch-Wets convergence. First we can prove the Kuratowski convergence of the minimal sets MinA_n in the setting of normed linear spaces ordered by a closed convex cone endowed with a base (without any compactness requirement). It is interesting to remark that in the last case the minimal sets MinA_n converge to the set of positive properly efficient points Pos$A$. By adding an assumption on the topological structure of the set Pos$A$, we obtain a stronger result: the Attouch-Wets convergence of the sets MinA_n to Pos$A$. Moreover, to obtain an approximation of every minimal point of A by minima of perturbations, we restrict to the case where A is a compact and convex set with respect to the ordering cone.

It is not surprising that proper efficiency plays an important role in stability in vector optimization. This notion was originally introduced in order to eliminate some "unstable" situations from the solution of a vector optimization problem. In the context of parametric problems in [12] properly efficient points are characterized as stable solutions of a family of suitable scalarizations.

2 Preliminaries

Let X be a normed linear space ordered by a pointed (i.e. $K \cap (-K) = \{0\}$) closed convex cone K. Let X^* be the topological dual space of X, we define the strict polar cone of K

$$K^{+i} = \{\phi \in X^* : \phi(x) > 0 \text{ for all } K \setminus \{0\}\}.$$

A cone K is said to have a *base* G if $G \subset X$ is convex, $0 \notin \mathrm{cl}G$ and $K = \{\lambda g : \lambda \in \mathbb{R}^+, g \in G\}$. A based cone is necessarily pointed and convex. If K is convex and K^{+i} is nonempty, then K is based and you can construct a base as follows:

$$G = \{x \in X : \phi(x) = 1\} \cap K.$$

Given a subset A of Y we define the set

$$\mathrm{Min}A = \{a \in A : A \cap (a - K) = \{a\}\},$$

the elements of the set Min$A$ are called the *minimal points* of A (with respect to the order given by the cone K).

The notion of minimality can be restricted to proper minimality, in order to avoid some indesiderable situations (for a survey on proper minimality in vector optimization see, e.g., [9]). Among the various definitions of properly minimal points appeared in the literature we consider the classical definition of positive proper minimality. We say that $a_0 \in A$ is a *positive proper minimal point* of A if there exists some $\phi \in K^{+i}$ such that $\phi(a) \geq \phi(a_0)$, for all $a \in A$. We denote the set of positive proper minimal points of A by Pos$A$.

Now we recall a property, well known in the theory of vector optimization, that plays an important role in our work. A subset A of X satisfies the *domination property* if, for every $a \in A$, there is $a_0 \in \mathrm{Min}A$ such that $a \in a_0 + K$ (see, e.g., [10]).

The various notions of convergence of a sequence of sets play a central role in our results (for a complete overview on this topic see [4]). We recall that, given a subsequence of subsets $\{A_n\}$ in X, the *Kuratowski lower* and *upper limits* are definied as

$$\begin{aligned} \mathrm{Li}\ A_n &= \left\{x \in X : x = \lim_{n\to\infty} x_n, x_n \in A_n, \text{ for all large } n\right\}, \\ \mathrm{Ls}\ A_n &= \left\{x \in X : x = \lim_{s\to\infty} x_s, x_s \in A_{n_s}, \{n_s\} \subset \mathbb{N}\right\}. \end{aligned}$$

If a subset A of X satisfies the conditions Ls $A_n \subset A \subset$ Li A_n, we say that the sequence $\{A_n\}$ converges to A in the sense of *Kuratowski* and we denote it by $A_n \xrightarrow{K} A$.

When we consider the limits in the weak topology on X rather than in the original topology induced by the norm, we denote the lower and the upper limits above by w-Li A_n and w-Ls A_n. When w-Ls $A_n \subset A \subset$ Li A_n we say that $\{A_n\}$ converges to A in the sense of *Mosco* ($A_n \xrightarrow{M} A$). It is easy to observe that $A_n \xrightarrow{M} A$ if and only if $A_n \xrightarrow{K} A$ both in the original and in the weak topology. The Mosco convergence is particular appropriate when we work with closed convex sets in a reflexive Banach space.

The last notion of set-convergence that we recall is the Attouch-Wets convergence (also known as bounded Hausdorff). To illustrate this notion we introduce some notations. Let B_ρ be the ball centered at 0 and with radius ρ. Given $x \in X$ and two nonempty subsets A and B of X, we have

$$\begin{aligned}
d\,(x, A) &= \inf_{a \in A} \|x - a\| \qquad (d(x, \varnothing) = \infty),\\
e(A, B) &= \sup_{a \in A} d(a, B) \qquad (e(\varnothing, A) = 0 \quad e(\varnothing, \varnothing) = 0, \quad e(A, \varnothing) = \infty),\\
e_\rho(A, B) &= e(A \cap B_\rho, B) \quad \text{and} \quad h_\rho(A, B) = \max\{e_\rho(A, B), e_\rho(B, A)\}\,.
\end{aligned}$$

We say that the sequence $\{A_n\}$ converges to A in the sense of *Attouch-Wets* when

$$\lim_{n \to \infty} h_\rho(A_n, A) = 0 \quad \text{for all } \rho > 0.$$

As usual, we can split this notion of convergenge into an upper part ($\lim_{n\to\infty} e_\rho(A_n, A) = 0$) and a lower part ($\lim_{n\to\infty} e_\rho(A, A_n) = 0$).

We recall that, if X is a finite-dimensional space, the three quoted notions of set-convergence coincide.

3 Convergence of minimal sets: Mosco convergence of perturbed sets

We begin to study the role of convexity assumption on the convergence of minimal sets in the special environment of reflexive Banach spaces.

Theorem 3.1 *Let X be a reflexive Banach space, K be a closed cone with a bounded base and $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of subsets of X. If*

1. *A_n is a closed convex set for every n,*
2. *A_n satisfies the domination property for every n,*

3. $A_n \xrightarrow{M} A$

then $\mathrm{Min}A \subset$ w-Li $\mathrm{Min}A_n$.

Proof. If $\mathrm{Min}A$ is empty the thesis follows trivially. If $\mathrm{Min}A \neq \emptyset$, we suppose to the contrary that there exists $a \in \mathrm{Min}A$ such that $a \notin$ w-Li $\mathrm{Min}A_n$. By the hypothesis 3. there exists a sequences $\{a_n\}$ such that $a_n \in A_n$ and $\lim_n a_n = a$. Since $a \notin$ w-Li $\mathrm{Min}A_n$ we can split the sequence $\{a_n\}$ into two subsequences $\{a_{n_j}\}$ and $\{a_{n_k}\}$, with $\{n_j\}_j \cup \{n_k\}_k = \mathbb{N}$, such that $a_{n_j} \in A_{n_j} \setminus \mathrm{Min}A_{n_j}$ and $a_{n_k} \in \mathrm{Min}A_{n_k}$. Now there exists a sequence $\{b_j\}$ such that $b_j \in \mathrm{Min}A_{n_j}$ and $a_{n_j} - b_j \in K$. We consider two distinct cases.

- The sequence $\{b_j\}$ is a bounded sequence, hence it has a weakly convergent subsequence $\{b_{j_s}\}$. Since the sequence of sets $\{A_n\}$ Mosco converges to A we have w-$\lim_s b_{j_s} = b \in A$, then $a_{n_{j_s}} - b_{j_s} \rightharpoonup a - b$ and $a - b \in K$ because K is weakly closed. Now, recalling that $a \notin$ w-Li $\mathrm{Min}A_n$, we can easily observe that we can always choose $\{b_{j_s}\}$ not converging to a. Then it is $a \neq b$. The last assertion contradicts the minimality of a.

- The sequence $\{b_j\}$ is unbounded, i.e. $\|b_j\| \to +\infty$ (at least for a subsequence). Since $b_j \in a_{n_j} - K$ and $\lim_j a_{n_j} = a$, we have that there exists a sequence $\{\varepsilon_j\} \subset \mathbb{R}^+$ with $\lim_j \varepsilon_j = 0$ such that

$$b_j \in a - K + B_{\varepsilon_j}.$$

Let

$$\left[a_{n_j}, b_j\right] := \left\{x \in X : x = \alpha a_{n_j} + (1-\alpha) b_j, 0 \leq \alpha \leq 1\right\},$$

then we have $\left[a_{n_j}, b_j\right] \subset A_n \cap (a - K + B_{\varepsilon_j})$.

Let G be a bounded base for K; then there exists a real positive number η such that

$$0 \notin \mathrm{cl}(G + B_\eta).$$

We can choose a sequence of real numbers $\{\alpha_j\}$, with $\alpha_j \in [0,1)$, such that $c_j = \alpha_j a_{n_j} + (1-\alpha_j) b_j$ and

$$c_j \in a + B_\eta - G.$$

Indeed it is $\alpha_j a_{n_j} + (1-\alpha_j) b_j = a + h_j - (1-\alpha_j) k_j$, where $k_j = a_{n_j} - b_j$ and $\lim_j h_j = 0$; since $\{b_j\}$ is unbounded, we choose α_j such that $(1-\alpha_j) k_j \in G$.

Since $\mathrm{cl}(a + B_\eta - G)$ is a closed, bounded and convex set, there exists a subsequence $\{c_{j_k}\}$ weakly converging to $c \in \mathrm{cl}(a + \eta B_\eta - G)$. It is $c \neq a$.

Since $c_{n_k} \in A_{n_k} \cap (a - K + \varepsilon_{n_k} B_{\varepsilon_{n_k}})$, it is

$$c \in A \cap (a - K)\,,$$

against the minimality of a.

■

It is well known (see e.g. [7]) that in reflexive normed spaces the assumption that the ordering cone K has bounded base is restrictive: for instance, the nonnegative orthants in l^p, $1 < p < \infty$, have not a bounded base. Nevertheless the following example shows that the previous hypothesis cannot be relaxed even in a separable Hilbert space.

Example 3.2 *Let $X = l^2$ endowed with the usual norm and let K be the nonnegative orthant. Let $\{e_n\}_{n\in\mathbb{N}}$ be the orthonormal base of X with $e_n = \left(0, ...0, \underset{n-th}{1}, 0, ...\right)$. We consider the sequence of sets $\{A_n\}_{n\in\mathbb{N}}$ defined as follows*

$$A_n = [-ne_n, 0]\,.$$

We observe that $A_n \xrightarrow{M} \{0\}$ but $\mathrm{Min}A_n = \{-ne_n\}$ *and* w-Li $\mathrm{Min}A_n = \emptyset$ *while* $A = \mathrm{Min}A = \{0\}$.

In the finite dimensional setting, the requirement that the ordering cone has a bounded base is not restrictive: a closed convex pointed cone has always a bounded base.

In this environement, since weak and strong convergence coincide, we obtain a stronger result on the convergence of minimal frontiers.

Corollary 3.3 *Let X be an euclidean space and let K be a closed convex pointed cone. If*

1. *A_n is a closed convex set for every n,*
2. *A_n satisfies the domination property for every n,*
3. *$A_n \xrightarrow{K} A$,*

then $\mathrm{Min}A \subset \mathrm{Li}\ \mathrm{Min}A_n$.

4 Convergence of minimal sets: Attouch-Wets convergence of perturbed sets

In Theorem 3.1 the thesis gives informations involving only the weak topology. In order to obtain meaningful results, in the sense of strong convergence or of quantitative metric estimations, we have to make different assumptions on the convergence of the perturbed sets A_n. Moreover, using a stronger notion of set convergence we can establish our results in a more general framework: X is a normed linear space and K is a closed convex cone such that $K^{+i} \neq \varnothing$. We remark that the last assumption on the ordering cone K is not unduly restrictive, see [7]. Finally we emphasize the fact that we obtain the convergence of MinA_n to the positive properly efficient frontiers of the set A.

The following lemma will be useful in the next theorem. We recall that, given two subsets E and F of X, the gap fuctional D is defined as follows:

$$D(E,F) = \inf_{e\in E, f\in F} \|e-f\| = \inf_{e\in E} d(e,F) = \inf_{f\in F} d(f,E).$$

Lemma 4.1 *Let X be a normed linear space, K be a closed convex cone with $K^{+i} \neq \varnothing$. Then for any $\phi \in K^{+i}$ the half-space $H^+ := \{x \in X : \phi(x) \geq 0\}$ satisfies , for every $\delta > 0$, the relation*

$$D\left(\left(-K\backslash B_\delta\right), H^+\right) > 0,$$

where D is the gap functional.

Proof. Let $G = \{x \in X : \phi(x) = 1\} \cap K$ be a base for K (see [7]). Then there exists a positive real number $\bar{\varepsilon}$ such that

$$-G + B_{\bar{\varepsilon}} \subset H^- = \{x \in X : \phi(x) < 0\}.$$

To the contrary, suppose that for any real number $\varepsilon > 0$ it is $-G+B_\varepsilon \cap H^+ \neq \emptyset$. Then a sequence $\{x_n\}$ exists such that $x_n \in -G+B_{\varepsilon_n} \cap H^+$,where $\varepsilon_n \to 0$. The last relation implies that $\phi(x_n) \to -1$ and $\liminf \phi(x_n) \geq 0$,that is absurd. Let us consider the cone $K_{\bar{\varepsilon}} = \mathrm{cl}(\mathrm{cone}\left(-G + B_{\bar{\varepsilon}}\right))$. Since $K_{\bar{\varepsilon}}$ is a closed and pointed cone (see e.g. [5]), it is $K_{\bar{\varepsilon}} \backslash \{0\} \subset H^-$and $-K\backslash \{0\} \subset$ int $K_{\bar{\varepsilon}}$. The thesis immediately follows. ■

Theorem 4.2 *Let X be a normed linear space, K be a closed convex cone with $K^{+i} \neq \varnothing$ and let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of subsets of X. If*

1. *A_n is a closed convex set for every n,*

2. A_n *satisfies the domination property for every* n,

3. $A_n \overset{AW}{\longrightarrow} A$,

then $\mathrm{Pos}A \subset \mathrm{Li}\ \mathrm{Min}A_n$.

Proof. If Pos$A$ is empty the theorem is trivial. If Pos$A \neq \emptyset$ we prove the thesis by contradiction. Let $a \in \mathrm{Pos}A$ such that $a \notin \mathrm{Li}\ \mathrm{Min}A_n$.

Without loss of generality, let $a = 0$. Since the lower part of Attouch-Wets convergence implies the lower Kuratowski convergence there exists a sequence $\{a_n\}$ such that $a_n \in A_n$ and $a_n \to 0$. Since $0 \notin \mathrm{Li}\ \mathrm{Min}A_n$ we can split the sequence $\{a_n\}$ into two subsequences $\{a_{n_j}\}$ and $\{a_{n_k}\}$, with $\{n_j\}_j \cup \{n_k\}_k = \mathbb{N}$, such that $a_{n_j} \in A_{n_j} \setminus \mathrm{Min}A_{n_j}$.and $a_{n_k} \in \mathrm{Min}A_{n_k}$. Now, by 2, there exists a sequence $\{b_j\}$ such that $b_j \in \mathrm{Min}A_{n_j}$ and $b_j \in a_{n_j} - K$. By 1, it is, for any $j \in \mathbb{N}$,

$$[a_{n_j}, b_j] \subset A_{n_j} \cap (a_{n_j} - K),$$

where $[a_{n_j}, b_j] = \{x \in X : x = \alpha a_{n_j} + (1-\alpha)b_j, \alpha \in [0,1]\}$.

Now, recalling that $a_n \to 0$ and using the upper part of the Attouch-Wets convergence, for any $n \in \mathbb{N}$ and for any $\rho > 0$, we have

$$[a_{n_j}, b_j] \cap B_\rho \subset (A + B_{\eta_n}) \cap (\eta_n B_{\eta_n} - K), \tag{1}$$

where $\{\eta_j\}$ is a sequence of real numbers such that $\eta_j \to 0$.

Since $0 \in \mathrm{Pos}A$, there exists $\phi \in K^{+i}$ such that

$$A \subset H^+ = \{x \in X : \phi(x) \geq 0\}.$$

Then from Lemma 4.1, it follows that $D((-K \backslash B_\delta), A) > 0$.This inequality implies that for every $\delta > 0$ there exists n_o such that for every $n \geq n_o$, it is

$$(A + \eta_n B_{\eta_n}) \cap (\eta_n B_{\eta_n} - K) \subset B_\delta. \tag{2}$$

Since $a \notin \mathrm{Li}\ \mathrm{Min}A_n$ the sequence $\{b_j\}$ has a subsequence not converging to a. Then we obtain a contradiction (against 1 and 2). ■

Remark 4.3 *The convergence of the sequence of sets* $\{A_n\}$ *effectively used in the proof of the previous theorem is weaker than the Attouch-Wets convergence. We need only the upper part of Attouch-Wets convergence while for the lower part Kuratowski convergence is enough. This type of convergence is known in the literature as b-proximal convergence (see, e.g., [15]).*

If we add a compactness assumption involving only the minimal set of A we obtain a stronger convergence of MinA_n to the minimal set of A.

Theorem 4.4 *Let X be a normed linear space, K be a closed convex cone with $K^{+i} \neq \varnothing$ and let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of subsets of X. If*

1. *A_n is a closed convex set for every n,*
2. *A_n satisfies the domination property for every n,*
3. *$A_n \overset{AW}{\to} A$,*
4. *$\mathrm{Pos}A \cap B_\rho$ is a compact set,*

then for each $\rho > 0$, $\lim_{n\to\infty} e_\rho\left(\mathrm{Pos}A, \mathrm{Min}A_n\right) = 0$.

Proof. If Pos$A$ is empty the theorem is trivial. If Pos$A \neq \emptyset$, let us suppose that the thesis of the theorem does not hold. Then there exist $\rho > 0$, $\varepsilon > 0$ and a subsequence $\{A'_k\}_{k\in\mathbb{N}}$ of $\{A_n\}$, such that

$$e_\rho(\mathrm{Pos}A, \mathrm{Min}A'_k) > 2\varepsilon$$

for every k. This fact implies that, for every k, there exists $m_k \in \mathrm{Pos}A \cap B_\rho$ such that

$$d(m_k, \mathrm{Min}A'_k) > 2\varepsilon.$$

Since $PosA \cap B_\rho$ is compact we can extract a convergent subsequence $\{m_{k_j}\}$ from $\{m_k\}$ .Clearly $m_{k_j} \to a \in PosA$. Without loss of generality we can consider $a = 0$. Since the lower part of Attouch-Wets convergence implies the lower Kuratowski convergence there exists, for every k, a sequence $\left\{a_s^{(k)}\right\}_{s\in\mathbb{N}}$ such that $a_s^{(k)} \in A'_s$ and $a_s^{(k)} \to m_k$. Now we consider the sequence $\left\{a_{k_j}^{(k_j)}\right\}_{j\in\mathbb{N}}$. Clearly this sequence satisfies the following properties

- $a_{k_j}^{(k_j)} \to 0,$
- $d(m_{k_j}, \mathrm{Min}A'_{k_j}) > \varepsilon$ at least definitively

Now we can follow the line of the proof of the previous theorem in order to reach a contradiction. ■

A well known theorem by Arrow, Barankin and Blackwell (1953) established the density of the set Pos$A$ in Min$A$,whenever A is a compact convex set of $\mathbb{R}^n$. Many generalizations of this result appeared in the literature (for a survey on this topic, see [6]). In order to obtain convergence to the set Min$A$, instead of Pos$A$, we use a recent extension of these results appeared in [17]: in a Banach space ordered by a normal closed convex cone with a base the set of positive minimal points is dense in Min$A$,whenever A is a K-convex and K-compact set. We recall that a set A is K-compact when any cover of A of the form $\{U_\alpha : \alpha \in I, U_\alpha \text{ are open}\}$ admits a finite subcover (see [10]). The requirement that A is K-convex is satisfied because A is convex since it is the limit of a sequence of convex sets A_n (see [1], ch. 3).

Of course, it possible to derive similar results with slightly different assumptions, by using other generalizations of Arrow-Barankin-Blackwell theorem.

Corollary 4.5 *Let X be a Banach space, K be a normal closed convex cone with $K^{+i} \neq \emptyset$ and let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of subsets of X. If*

1. *A_n is a closed convex set for every n,*
2. *A_n satisfies the domination property for every n,*
3. *$A_n \overset{AW}{\to} A$,*
4. *A is a K-compact and set,*

then $\mathrm{Min}A \subset \mathrm{Li}\ \mathrm{Min}A_n$.

Corollary 4.6 *Let X be a Banach space, K be a normal closed convex cone with $K^{+i} \neq \emptyset$ and let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of subsets of X. If*

1. *A_n is a closed convex set for every n,*
2. *A_n satisfies the domination property for every n,*
3. *$A_n \overset{AW}{\to} A$,*
4. *$\mathrm{Pos}A \cap B_\rho$ is a compact set,*
5. *A is a K-compact and set,*

then for each $\rho > 0$, $\lim_{n\to\infty} e_\rho(\mathrm{Min}A, \mathrm{Min}A_n) = 0$.

The proofs ot these corollaries follows easily recalling that the lower convergence (in the sense of Kuratowski or Attouch-Wets) to a set A is equivalent to the lower convergence to the closure of A.

5 Comparison with the existing results

Among the results on the stability of minimal sets we focus on those that can be formulated in terms of the lower part of some notion of set convergence. In this section we present some existing theorems, in order to make a comparison with our results. We state these results as in [11].

Theorem 5.1 *([13], [8]) Let X be a normed space and K be a closed convex cone. Assume that the following conditions hold*

1. $A_n \xrightarrow{K} A$,
2. A_n *satisfies the domination property for every* n,
3. *if* $a_n \in A_n$ *is such that* $\lim_{n \to +\infty} a_n$ *exists and* $e_n \in \mathrm{Min}A_n \cap (a_n - K)$, *then admits a convergent subsequence,*

then $\mathrm{Min}A \subset \mathrm{Li}\ \mathrm{Min}A_n$.

If we consider a Banach spaces we have the following results.

Theorem 5.2 *([2]) Let X be a Banach space, K is such that*

$$-K \subset \{x \in X : l(x) + \varepsilon \|x\| \leq 0\}$$

for some $l \in X^*, \varepsilon > 0$ and let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence of subsets of X. If

1. $A_n \xrightarrow{K} A$,
2. $\inf_{n \in \mathbb{N}} \inf_{x \in A_n} l(x) > -\infty$,
3. *for every $\rho > 0$, there exists a compact subset $C_\rho \subset X$, such that, for every $n \in \mathbb{N}$, $\mathrm{Min}A_n \cap B_\rho \subset C_\rho$,*

then $\mathrm{Min}A \subset \mathrm{Li}\ \mathrm{Min}A_n$.

The original version of this theorem gives also an existence condition for the minimal point under the same hypothesis. We underline that the assumptions made about the ordering cone K imply that K has a bounded base. In fact, we recall that every subset of a real normed space with the form

$$\{x \in X : l(x) + \varepsilon \|x\| \leq 0\}$$

where $l \in X^*$ and $\varepsilon > 0$, is a pointed closed convex cone with a bounded base (if it is not trivial) that is called Bishop-Phelps cone, see [14]. In a reflexive Banach space the compactess assumption 3. can be dropped using Mosco convergence of $\{A_n\}$ to A, instead of Kuratowski convergence (see [2], Theorem 3.5). We recall also that in this setting the boundedness of the base implies its weak compactness.

Theorem 5.3 *([2]) Let X be a reflexive Banach space, K be a closed convex cone such that*

$$-K \subset \{x \in X : l(x) + \varepsilon \|x\| \leq 0\}$$

for some $l \in X^*, \varepsilon > 0$ and let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence of subsets of X. If

1. $A_n \xrightarrow{M} A$,
2. $\inf_{n \in \mathbb{N}} \inf_{x \in A_n} l(x) > -\infty$,

then $\mathrm{Min}A \subset \mathrm{Li}\ \mathrm{Min}A_n$.

The last result that we quote gives a stronger convergence of minimal sets, i.e. it provides some conditions to assure the Attouch-Wets convergence of minimal sets of A_n to $\mathrm{Min}A$.

Theorem 5.4 *([11]) Assume that the following conditions hold*

1. $A_n \xrightarrow{K} A$,
2. *A_n satisfies the domination property for every n,*
3. *if $a_n \in A_n$ is such that $\lim\limits_{n \to +\infty} a_n$ exists and $e_n \in \mathrm{Min}A_n \cap (a_n - K)$, then admits a convergent subsequence,*
4. *for every $\rho > 0$, $\mathrm{Min}A \cap B_\rho$ is relatively compact.*

Then for each $\rho > 0$, $\lim_{n \to \infty} e_\rho (\mathrm{Min}A, \mathrm{Min}A_n) = 0$.

We recall that in theorems 5.1 and 5.4 the authors take into account also perturbations of the ordering cone. We just consider the case of a constant cone K, for the sake of easier comparisons with our results.

The hypotheses used in the present work have a completely different nature. The results quoted above is based on some "compactness" requirement of the entire sequence $\{A_n\}$ (Hypotesis 3. in Theorem 5.1, 3. in theorem 5.2 or 3. in Theorem 5.4) and also in Theorem 5.3, where there is not a compactness assumption, the bounddedness condition 2. concerns the whole sequence $\{A_n\}$. We avoid any requirement involving the family $\{A_n\}_{n\in\mathbb{N}}$ as a whole by the use of a convexity assumption on each set A_n. We show by some examples that, even in a euclidean space, the two approaches are distinct.

The following example proves that, even in $\mathbb{R}^n$, our assumptions do not imply the compactness assumption 3 of theorem 5.1 and 5.4.

Example 5.5 *Let $X = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$. Let*

$$A_n = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = -1/n, x_2 \geq -n\}.$$

A_n is a closed convex set that satisfies the dominance property, for every n. It is $A_n \xrightarrow{K} A = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0\}$ (hence $A_n \xrightarrow{AW} A$). We can see that condition 3. in 5.1 does not hold. As a matter of fact we consider the sequence $\{a_n\} = \{(-1/n, 0)\}$. It is $\lim_{n\to+\infty} a_n = (0,0)$ and the only sequence $\{e_n\}$ such that $e_n \in \mathrm{Min} A_n \cap (a_n - K)$ is the sequence $\{(-1/n, -n)\}$ that has not any convergent subsequence.

The following example shows that our assumptions do not imply the boundedness condition 2 of 5.2 and 5.3.

Example 5.6 *Let $X = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$. Let*

$$A_n = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = (-1 + 1/n)x_1\}.$$

A_n is a closed convex set that satisfies the dominance property, for every n.It is $A_n \xrightarrow{K} A = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = -x_1\}$. It can be easily verifies that

$$\inf_{n\in\mathbb{N}} \inf_{x\in A_n} l(x) = -\infty$$

for every linear functional l.

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