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Second-order conditions in $C^{1,1}$ constrained vector optimization

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Abstract

We consider the constrained vector optimization problem $\min_C f(x),\ g(x) \in -K$, where $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are $C^{1,1}$ functions, and $C \subset \mathbb{R}^m$ and $K \subset \mathbb{R}^p$ are closed convex cones with nonempty interiors. Two type of solutions are important for our consideration, namely w-minimizers (weakly efficient points) and i-minimizers (isolated minimizers). We formulate and prove in terms of the Dini directional derivative second-order necessary conditions a point x^0 to be a w-minimizer and second-order sufficient conditions x^0 to be a i-minimizer of order two. We discuss the possible reversal of the sufficient conditions under suitable constraint qualifications of Kuhn-Tucker type. The obtained results improve the ones in Liu, Neittaanmäki, Křířek [20].

Key words: Vector optimization, $C^{1,1}$ optimization, Dini derivatives, Second-order conditions, Duality.

Math. Subject Classification: 90C29, 90C30, 90C46, 49J52.

1 Introduction

In this paper we deal with the constrained vector optimization problem

$$\min_C f(x) , \quad g(x) \in -K , \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are given functions, and $C \subset \mathbb{R}^m$ and $K \subset \mathbb{R}^p$ are closed convex cones with nonempty interiors. Here n, m and p are positive integers. In the case when f and g are $C^{1,1}$ functions we derive second-order optimality conditions a point x^0 to be a solution of this problem. The paper is thought as a continuation of the investigation initiated by the authors in [7], where first-order conditions are derived in the case when f and g are $C^{0,1}$ functions. Recall that a function is said to be $C^{k,0}$ if it is k-times Fréchet differentiable with locally Lipschitz k-th derivative. The $C^{0,1}$ functions are the locally Lipschitz functions. The $C^{1,1}$ functions have been introduced in Hiriart-Urruty, Strodiot, Hien Nguen [13] and since then have found various application in optimization. In particular second-order conditions for $C^{1,1}$ scalar problems are studied in [13, 5, 16, 25, 26]. Second-order optimality conditions in vector optimization are investigated in [1, 4, 15, 21, 24], and what concerns $C^{1,1}$ vector optimization in [9, 10, 18, 19, 20]. The given in the present paper approach and results generalize that of [20].

The assumption that f and g are defined on the whole space $\mathbb{R}^n$ is taken for convenience. Since we deal only with local solutions of problem (1), evidently our results generalize straightforward for functions f and g being defined on an open subset of $\mathbb{R}^n$. Usually the solutions of (1) are called points of efficiency. We prefer, like in the scalar optimization, to call them minimizers. In Section 2 we define different type of minimizers. Among them in our considerations an important role play the w-minimizers (weakly efficient points) and the i-minimizers (isolated minimizers). When we say first or second-order conditions we mean as usually conditions expressed in suitable first or second-order derivatives of the given functions. Here we deal with the Dini directional derivatives. In Section 3 first-order Dini derivative is defined and after [7] first-order optimality conditions for $C^{0,1}$ functions are recalled. In Section 4 we define second-order Dini derivative and formulate and prove second-order optimality conditions for $C^{1,1}$ functions. We note that our results improve the ones obtained in Liu, Neittaanmäki, Křížek [20]. In Section 5 we discuss the possibility to revert the obtained sufficient conditions under suitable constraint qualifications.

2 Preliminaries

For the norm and the dual parity in the considered finite-dimensional spaces we write $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$. From the context it should be clear to exactly which spaces these notations are applied.

For the cone $M \subset \mathbb{R}^k$ its positive polar cone M' is defined by $M' = \{\zeta \in \mathbb{R}^k \mid \langle\zeta,\,\phi\rangle \geq 0 \text{ for all } \phi \in M\}$. The cone M' is closed and convex. It is known that $M'' := (M')' = \operatorname{cl}\operatorname{co} M$, see Rockafellar [23, Theorem 14.1, page 121]. In particular for the closed convex cone M we have $M' = \{\zeta \in \mathbb{R}^k \mid \langle\zeta,\,\phi\rangle \geq 0 \text{ for all } \phi \in M\}$ and $M = M'' = \{\phi \in \mathbb{R}^k \mid \langle\zeta,\,\phi\rangle \geq 0 \text{ for all } \zeta \in M'\}$.

If $\phi \in -\operatorname{cl}\operatorname{conv} M$, then $\langle\zeta,\,\phi\rangle \leq 0$ for all $\zeta \in M'$. We set $M'[\phi] = \{\zeta \in M' \mid \langle\zeta,\,\phi\rangle = 0\}$. Then $M'[\phi]$ is a closed convex cone and $M'[\phi] \subset M'$. Consequently its positive polar cone $M[\phi] = (M'[\phi])'$ is a closed convex cone, $M \subset M[\phi]$ and its positive polar cone satisfies $(M[\phi])' = M'[\phi]$. In this paper we apply this notation for $M = K$ and $\phi = g(x^0)$. Then we deal with the cones $K'[g(x^0)]$ (we call this cone the index set of problem (1) at x^0) and $K[g(x^0)]$.

For the closed convex cone M' we apply in the sequel the notation $\Gamma_{M'} = \{\zeta \in M' \mid \|\zeta\| = 1\}$. The sets $\Gamma_{M'}$ is compact, since it is closed and bounded.

Now we recall the concept of the so called oriented distance from a point to a set, being applied to give a scalar characterization of some type of solutions of vector optimization problems. Given a set $A \subset \mathbb{R}^k$, then the distance from $y \in \mathbb{R}^k$ to A is $d(y, A) = \inf\{\|a - y\| \mid a \in A\}$. The oriented distance from y to A is defined by $D(y, A) = d(y, A) - d(y, \mathbb{R}^k \setminus A)$. The function D is introduced in Hiriart-Urruty [11, 12] and since then many authors show its usefulness in optimization. Ginchev, Hoffmann [8] apply the oriented distance to study approximation of set-valued functions by single-valued ones and in case of a convex set A show the representation $D(y, A) = \sup_{\|\xi\|=1} \left(\langle\xi,\,y\rangle - \sup_{a\in A}\langle\xi,\,a\rangle\right)$ (this formula resembles [23] the conjugate of the support function of A). From here, when $A = -C$ and C is a closed convex cone, taking into account

$$\sup_{a\in -C}\langle\xi,\,a\rangle = \begin{cases} 0 & , \quad \xi \in C', \\ +\infty & , \quad \xi \notin C', \end{cases}$$

we get easily $D(y, -C) = \sup_{\|\xi\|=1,\,\xi\in C'}\langle\xi,\,y\rangle$.

In terms of the distance function we have

$$K[g(x^0)] = \{w \in \mathbb{R}^p \mid \limsup_{t\to 0^+} \frac{1}{t}\, d(g(x^0) + tw, -C) = 0\},$$

that is $K[g(x^0)]$ is the contingent cone [3] of K at $g(x^0)$.

We call the solutions of problem (1) minimizers. The solutions are understood in a local sense. In any case a solution is a feasible point x^0, that is a point satisfying the constraint $x^0 \in -K$.

The feasible point x^0 is said to be a w-minimizer (weakly efficient point) for (1) if there exists a neighbourhood U of x^0, such that $f(x) \notin f(x^0) - \operatorname{int} C$ for all $x \in U \cap g^{-1}(-K)$. The feasible point x^0 is said to be a e-minimizer (efficient point) for (1) if there exists a neighbourhood U of x^0, such that

$f(x) \notin f(x^0) - (C \setminus \{0\}$ for all $x \in U \cap g^{-1}(-K)$. We say that the feasible point x^0 is a s-minimizer (strong minimizer) for (1) if there exists a neighbourhood U of x^0, such that $f(x) \notin f(x^0) - C$ for all $x \in (U \setminus \{x^0\}) \cap g^{-1}(-K)$.

In [7] through the oriented distance the following characterization is derived. The feasible point x^0 is a w-minimizer (s-minimizer) for the vector problem (1) if and only if x^0 is a minimizer (strong minimizer) for the scalar problem

$$\min \, D(f(x) - f(x^0), -C) \,, \quad g(x) \in -K \,. \tag{2}$$

Here we have an example of a scalarization of the vector optimization problem. By scalarization we mean a reduction of the vector problem to an equivalent scalar optimization problem.

We introduce another concept of optimality. We say that the feasible point x^0 is a i-minimizer (isolated minimizer) of order k, $k > 0$, for (1) if there exists a neighbourhood U of x^0 and a constant $A > 0$ such that $D(f(x) - f(x^0), -C) \geq A \, \|x - x^0\|^k$ for $x \in U \cap g^{-1}(-K)$.

In spite that the notion of isolated minimizer is defined through the norms, the following reasoning shows that in fact it is norm-independent. From the nonnegativeness of the right-hand side the definition of the i-minimizer of order k equivalently can be given by the inequality $d(f(x) - f(x^0), -C) \geq A \, \|x - x^0\|^k$ for $x \in U \cap g^{-1}(-K)$. Assume that another pair of norms in the original and image space denoted $\|\cdot\|$ is introduced. As it is known any two norms in a finite dimensional space over the reals are equivalent. Therefore there exist positive constants α_1, α_2, β_1, β_2, such that

$$\alpha_1 \|x\| \leq \|x\|_1 \leq \alpha_2 \|x\| \quad \text{for all} \quad x \in X := \mathbb{R}^n \,,$$

$$\beta_1 \|y\| \leq \|y\|_1 \leq \beta_2 \|y\| \quad \text{for all} \quad y \in Y := \mathbb{R}^m \,.$$

Denote by d^1 the distance associated to the norm $\|\cdot\|_1$. For $A \subset Y$ we have

$$d^1(y, A) = \sup\{\|y - a\| \mid a \in A\} \geq \sup\{\beta_1 \|y - a\| \mid a \in A\} = \beta_1 d(y, A) \,,$$

whence

$$d^1(f(x) - f(x^0), -C) \geq \beta_1 d(f(x) - f(x^0), -C)$$

$$\geq \beta_1 A \, \|x - x^0\|^k \geq \frac{\beta_1 A}{\alpha_2^k} \, \|x - x^0\|_1^k \;\text{ for }\; x \in U \cap g^{-1}(-K) \,.$$

Therefore, if x^0 is a i-minimizer of order k for (1) with respect to the pair of norms $\|\cdot\|$, it is also a i-minimizer with respect to the pair of norms $\|\cdot\|_1$, only the constant A should be changed to $(\beta_1 A)/\alpha_2^k$. Obviously, each i-minimizer is a s-minimizer. Further each s-minimizer is a e-minimizer and each e-minimizer is a w-minimizer (claiming this we assume that $C \neq \mathbb{R}^m$).

The concept of an isolated minimizer for scalar problems is introduced in Auslender [2]. For vector problems it has been extended in Ginchev [6], Ginchev, Guerraggio, Rocca [7], and under the name of strict minimizers in Jiménez [14] and Jiménez, Novo [15]. We prefer the name "isolated minimizer" given originally by A. Auslender.

3 First-order conditions

In this section after [7] we recall optimality conditions for problem (1) with $C^{0,1}$ data in terms of the first-order Dini directional derivative. The proofs can be found in [7].

Given a $C^{0,1}$ function $\Phi : \mathbb{R}^n \to \mathbb{R}^k$, we define the Dini directional derivative $\Phi'_u(x^0)$ of Φ at x^0 in direction $u \in \mathbb{R}^n$ as the set of the cluster points of $(1/t)(\Phi(x^0 + tu) - \Phi(x^0))$ as $t \to 0^+$, that is as the upper limit

$$\Phi'_u(x^0) = \operatorname*{Limsup}_{t \to 0^+} \frac{1}{t} \left(\Phi(x^0 + tu) - \Phi(x^0) \right) \,.$$

If Φ is Fréchet differentiable at x^0 then the Dini derivative is a singleton, coincides with the usual directional derivative and can be expressed in terms of the Fréchet derivative $\Phi'(x^0)$ by $\Phi'_u(x^0) = \Phi'(x^0)u$. In connection with problem (1) we deal with the Dini derivative of the function $\Phi : \mathbb{R}^n \to \mathbb{R}^{m+p}$, $\Phi(x) = (f(x),\, g(x))$. Then we use the notation $\Phi'_u(x^0) = (f(x^0),\, g(x^0))'_u$. If at least one of the derivatives $f'_u(x^0)$ and $g'_u(x^0)$ is a singleton, then $(f(x^0), g(x^0))'_u = (f'_u(x^0), g'_u(x^0))$. Let us turn attention that always $(f(x^0),\, g(x^0))'_u \subset f'_u(x^0) \times g'_u(x^0)$, but in general these two sets do not coincide.

The following constraint qualification appears in the Sufficient Conditions part of Theorem 1.

$$\mathbb{Q}_{0,1}(x^0) \; : \qquad \begin{array}{c} \text{If } g(x^0) \in -K \text{ and } \dfrac{1}{t_k}\left(g(x^0 + t_k u^0) - g(x^0)\right) \to z^0 \in -K(x^0) \\ \text{then } \exists\, u^k \to u^0 : \exists\, k_0 \in \mathbb{N} : \forall\, k > k_0 : g(x^0 + t_k u^k) \in -K\,. \end{array}$$

The constraint qualification $\mathbb{Q}_{0,1}(x^0)$ is of Kuhn-Tucker type. In 1951 Kuhn, Tucker [17] published the classical variant for differentiable functions and since then it is often cited in optimization theory.

Theorem 1 (First-order conditions) *Consider problem (1) with f, g being $C^{0,1}$ functions and C and K closed convex cones.*

(Necessary Conditions) *Let x^0 be w-minimizer of problem (1). Then for each $u \in S$ the following condition is satisfied:*

$$\mathbb{N}'_{0,1} \; : \qquad \begin{array}{c} \forall (y^0,\, z^0) \in (f(x^0),\, g(x^0))'_u : \exists (\xi^0,\, \eta^0) \in C' \times K' : \\ (\xi^0,\, \eta^0) \neq (0,\, 0), \quad \langle \eta^0,\, g(x^0) \rangle = 0 \quad \textit{and} \quad \langle \xi^0,\, y^0 \rangle + \langle \eta^0,\, z^0 \rangle \geq 0\,. \end{array}$$

(Sufficient Conditions) *Let $x^0 \in \mathbb{R}^n$ and suppose that for each $u \in S$ the following condition is satisfied:*

$$\mathbb{S}'_{0,1} \; : \qquad \begin{array}{c} \forall (y^0,\, z^0) \in (f(x^0),\, g(x^0))'_u : \exists (\xi^0,\, \eta^0) \in C' \times K' : \\ (\xi^0,\, \eta^0) \neq (0,\, 0), \quad \langle \eta^0,\, g(x^0) \rangle = 0 \quad \textit{and} \quad \langle \xi^0,\, y^0 \rangle + \langle \eta^0,\, z^0 \rangle > 0\,. \end{array}$$

Then x^0 is a i-minimizer of order one for problem (1).

Conversely, if x^0 is a i-minimizer of order one for problem (1) and the constraint qualification $\mathbb{Q}_{0,1}(x^0)$ holds, then condition $\mathbb{S}'_{0,1}$ is satisfied.

If g is Fréchet differentiable at x^0, then instead of constraint qualification $\mathbb{Q}_{0,1}(x^0)$ we may consider the constraint qualification $\mathbb{Q}_1(x^0)$ given below.

$$\mathbb{Q}_1(x^0) \; : \qquad \begin{array}{c} \text{If } g(x^0) \in -K \text{ and } g'(x^0)u^0 = z^0 \in -K(x^0) \text{ then} \\ \text{there exists } \delta > 0 \text{ and a differentiable injective function} \\ \varphi : [0,\, \delta] \to -K \text{ such that } \varphi(0) = x^0 \text{ and } \varphi'(0) = g'(x^0)u^0\,. \end{array}$$

In the case of a polyhedral cone K in $\mathbb{Q}_1(x^0)$ the requirement $\varphi : [0,\, \delta] \to -K$ can be replaced by $\varphi : [0,\, \delta] \to -K(x^0)$. This condition coincides with the classical Kuhn-Tucker constraint qualification (compare with Mangasarian [22, p. 102]).

The next theorem is a reformulation of Theorem 1 for C^1 problems, that is problems with f and g being C^1 functions.

Theorem 2 *Consider problem (1) with f, g being C^1 functions and C and K closed convex cones.*

(Necessary Conditions) *Let x^0 be w-minimizer of problem (1). Then for each $u \in S$ the following condition is satisfied:*

$$\mathbb{N}'_1 \; : \qquad \begin{array}{c} \exists (\xi^0,\, \eta^0) \in C' \times K' \setminus \{(0,\, 0)\} : \\ \langle \eta^0,\, g(x^0) \rangle = 0 \quad \textit{and} \quad \langle \xi^0,\, f'(x^0)u \rangle + \langle \eta^0,\, g'(x^0)u \rangle \geq 0\,. \end{array}$$

(Sufficient Conditions) *Let $x^0 \in \mathbb{R}^n$.*

Suppose that for each $u \in S$ the following condition is satisfied:

$$\mathbb{S}_1' : \quad \begin{array}{c} \exists(\xi^0,\, \eta^0) \in C' \times K' \setminus \{(0,\,0)\} : \\ \langle \eta^0,\, g(x^0)\rangle = 0 \quad \text{and} \quad \langle \xi^0,\, f'(x^0)u\rangle + \langle \eta^0,\, g'(x^0)u\rangle > 0\,. \end{array}$$

Then x^0 is a i-minimizer of first order for problem (1).

Conversely, if x^0 is a i-minimizer of first order for problem (1) and let the constraint qualification $\mathbb{Q}_1(x^0)$ have place, then condition $\mathbb{S}_1'$ is satisfied.

The pairs of vectors $(\xi^0,\, \eta^0)$ are usually referred to as the Lagrange multipliers. Here we have different Lagrange multipliers to different $u \in S$ (and different $(y^0,\, z^0) \in (f(x^0), g(x^0))'_u$). The natural question arises, whether a common pair $(\xi^0,\, \eta^0)$ can be chosen to all directions. The next example shows that the answer is negative even for C^1 problems.

Example 1 *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = (x_1,\, x_1^2 + x_2^2)$, and $g : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = (x_1,\, x_2)$. Define $C = \{y \in (y_1,\, y_2) \in \mathbb{R}^2 \mid y_2 = 0\}$, $K = \mathbb{R}^2$. Then f and g are C^1 functions and the point $x^0 = (0,\,0)$ is a w-minimizer of problem (1) (in fact x^0 is also a i-minimizer of order two, but not a i-minimizer of order one). At the same time the only pair $(\xi^0,\, \eta^0) \in C' \times K'$ for which $\langle \xi^0,\, f'(x^0)u\rangle + \langle \eta^0,\, g'(x^0)u\rangle \geq 0$ for all $u \in S$ is $\xi^0 = (0,\,0)$ and $\eta^0 = (0,\,0)$.*

Theorem below 3 guarantees, that in the case of cones with nonempty interiors a nonzero pair $(\xi^0,\, \eta^0)$ exists, which satisfies the Necessary Conditions of Theorem 1 and which is common for all directions. This is the reason why the second-order conditions in the next section are derived under the assumption of C and K closed convex cones with nonempty interior.

Theorem 3 (Necessary Conditions) *Consider problem (1) with f, g being C^1 functions and C and K closed convex cones with nonempty interiors. Let x^0 be w-minimizer of problem (1). Then there exists a pair $(\xi^0,\, \eta^0) \in C' \times K' \setminus \{(0,\,0)\}$ such that $\langle \eta^0,\, g(x^0)\rangle = 0$ and $\langle \xi^0,\, f'(x^0)u\rangle + \langle \eta^0,\, g'(x^0)u\rangle = 0$ for all $u \in \mathbb{R}^n$. The latter equality could be written also as $\langle \xi^0,\, f'(x^0)\rangle + \langle \eta^0,\, g'(x^0)\rangle = 0$.*

4 Second-order conditions

In this section we derive optimality conditions for problem (1) with $C^{1,1}$ data in terms of the second-order Dini directional derivative.

Given a $C^{1,1}$ function $\Phi : \mathbb{R}^n \to \mathbb{R}^k$, we define the second-order Dini directional derivative $\Phi''_u(x^0)$ of Φ at x^0 in direction $u \in \mathbb{R}^n$ as the set of the cluster points of $(1/t^2)(\Phi(x^0 + tu) - \Phi(x^0) - \Phi'(x^0)u)$ as $t \to 0^+$, that is as the upper limit

$$\Phi''_u(x^0) = \operatorname*{Limsup}_{t \to 0^+} \frac{2}{t^2} \left(\Phi(x^0 + tu) - \Phi(x^0) - t\,\Phi'(x^0)u \right) .$$

If Φ is twice Fréchet differentiable at x^0 then the Dini derivative is a singleton and can be expressed in terms of the Hessian $\Phi''_u(x^0) = \Phi''(x^0)(u, u)$. In connection with problem (1) we deal with the Dini derivative of the function $\Phi : \mathbb{R}^n \to \mathbb{R}^{m+p}$, $\Phi(x) = (f(x),\, g(x))$. Then we use the notation $\Phi''_u(x^0) = (f(x^0),\, g(x^0))''_u$. Let us turn attention that always $(f(x^0),\, g(x^0))''_u \subset f''_u(x^0) \times g''_u(x^0)$, but in general these two sets do not coincide.

We need the following lemma.

Lemma 1 *Let $\Phi : \mathbb{R}^n \to \mathbb{R}^k$ be a $C^{1,1}$ function and Φ' be Lipschitz with constant L on the ball $\{x \mid \|x - x^0\| \leq r\}$, where $x^0 \in \mathbb{R}^n$ and $r > 0$. Then, for $u, v \in \mathbb{R}^m$ and $0 < t < r/\max(\|u\|, \|v\|)$ we have*

$$\left\| \frac{2}{t^2} \left(\Phi(x^0 + tv) - \Phi(x^0) - t\Phi'(x^0)v \right) - \frac{2}{t^2} \left(\Phi(x^0 + tu) - \Phi(x^0) - t\Phi'(x^0)u \right) \right\|$$

$$\leq L\,(\|u\|+\|v\|)\,\|v-u\|\,.$$

In particular, for $v=0$ we get

$$\left\|\frac{2}{t^2}\left(\Phi(x^0+tu)-\Phi(x^0)-t\Phi'(x^0)u\right)\right\|\leq L\,\|u\|^2\,.$$

Proof. For $0<t<r/\max(\|u\|,\|v\|)$ we have

$$\begin{aligned}\left\|\frac{2}{t^2}\left(\Phi(x^0+tv)-\Phi(x^0)-t\Phi'(x^0)v\right)-\frac{2}{t^2}\left(\Phi(x^0+tu)-\Phi(x^0)-t\Phi'(x^0)u\right)\right\|\\ =\frac{2}{t^2}\,\left\|\left(\Phi(x^0+tv)-\Phi(x^0+tu)\right)-t\Phi'(x^0)(v-u)\right\|\\ =\frac{2}{t}\,\left\|\int_0^1\left(\Phi'(x^0+tu+st(v-u))-\Phi'(x^0)\right)(v-u)\,ds\right\|\\ \leq 2L\int_0^1\|(1-s)u+sv\|\,ds\,\|v-u\|\\ \leq 2L\int_0^1((1-s)\|u\|+s\|v\|)\,ds\,\|v-u\|=L\,(\|u\|+\|v\|)\,\|v-u\|\,.\end{aligned}$$

□

The next theorem states second-order necessary conditions in primal form, that is in terms of directional derivatives.

Theorem 4 (Second-order necessary conditions, Primal form) *Consider problem (1) with f, g being $C^{1,1}$ functions, and C and K closed convex cones. Let x^0 be a w-minimizer for (1). Then for each $u\in\mathbb{R}^n$ the following two conditions hold.*

$\mathbb{N}_p'$: $\quad (f'(x^0)u,\; g'(x^0)u)\notin-(\operatorname{int}C\times\operatorname{int}K[g(x^0)])\,,$

$\mathbb{N}_p''$: *if* $(f'(x^0)u,\; g'(x^0)u)\in-(C\times K[g(x^0)]\setminus\operatorname{int}C\times\operatorname{int}K[g(x^0)])$
and $(y^0,z^0)\in(f(x^0),g(x^0))''_u$ *then it holds*
$$\operatorname{conv}\{(y^0,z^0),\;\operatorname{im}(f'(x^0),g'(x^0))\}\cap(-(\operatorname{int}C\times K[g(x^0)]))=\emptyset\,.$$

Proof. Condition $\mathbb{N}_p'$. Assume in the contrary, that

$$f'(x^0)u=\lim_{t\to0^+}\frac{1}{t}\left(f(x^0+tu)-f(x^0)\right)\in-\operatorname{int}C\quad\text{and}\tag{3}$$

$$g'(x^0)u=\lim_{t\to0^+}\frac{1}{t}\left(g(x^0+tu)-g(x^0)\right)\in-\operatorname{int}K[g(x^0)]\,.\tag{4}$$

We show first that there exists $\delta>0$ such that x^0+tu is feasible for $0<t<\delta$. For each $\bar\eta\in\Gamma_{K'}$ there exists $\delta\bar\eta$ and a neighbourhood $V(\bar\eta)$ of $\bar\eta$ such that $\langle\eta,\;g(x^0+tu)\rangle<0$ for $\eta\in V(\bar\eta)$. To show this we consider the two possibilities. If $\bar\eta\in\Gamma_{K[g(x^0)]'}$ the claim follows from

$$\frac{1}{t}\langle\bar\eta,\;g(x^0+tu)\rangle=\langle\bar\eta,\;\frac{1}{t}\left(g(x^0+tu)-g(x^0)\right)\rangle\to\langle\bar\eta,\;z^0\rangle<0\,.$$

For $\bar\eta\in\Gamma_{K'}\setminus\Gamma_{K[g(x^0)]'}$ we have $\langle\bar\eta,\;g(x^0)\rangle$, whence from the continuity of g we get $\langle\eta,\;g(x^0+tu)\rangle<0$ for (η,t) sufficiently close to $(\bar\eta,0)$. From the compactness of $\Gamma_{K'}$ it holds $\Gamma_{K'}\subset V(\bar\eta_1)\cup\ldots\cup V(\bar\eta_k)$. Then $\langle\eta,\;g(x^0+tu)\rangle<0$ holds for all $\eta\in\Gamma_{K'}$ with $0<t<\delta:=\min_{1\leq i\leq k}\delta(\bar\eta_i)$.
Now (3) gives that $f(x^0+tu)-f(x^0)\in-\operatorname{int}C$ for all sufficiently small t, which contradicts the assumption that x^0 is a w-minimizer for (1).

Condition $\mathbb{N}_p''$**.** Assume in the contrary, that for some $u \neq 0$ condition $\mathbb{N}_p''$ (for $u = 0$ obviously $\mathbb{N}_p''$ is satisfied). Therefore we have $(f'(x^0)u,\, g'(x^0)u) \in -(C \times K[g(x^0)] \setminus \operatorname{int} C \times \operatorname{int} K[g(x^0)])$ and there exists $(y(u), z(u)) \in (f(x^0), g(x^0))''_u$ such that for some $\lambda \in (0,\, 1)$ and some $w \in \mathbb{R}^n$ it holds

$$(1-\lambda)\, y(u) + \lambda\, f'(x^0)w \in -\operatorname{int} C\,,$$

$$(1-\lambda)\, z(u) + \lambda\, g'(x^0)w \in -\operatorname{int} K[g(x^0]\,.$$

According to the definition of the Dini derivative there exists a sequence $t_k \to 0^+$, such that $\lim_k y^k(u) = y(u)$ and $\lim_k z^k(u) = z(u)$, where for $v \in \mathbb{R}^n$ we put

$$y^k(v) = \frac{2}{t_k^2}\left(f(x^0 + t_k v) - f(x^0) - t_k f'(x^0)v\right)\,,$$

$$z^k(v) = \frac{2}{t_k^2}\left(g(x^0 + t_k v) - g(x^0) - t_k g'(x^0)v\right)\,.$$

Let $v^k \to u$ and f and g be Lipschitz with constant L in a neighbourhood of x^0. For k "large enough" we have, as in Lemma 1,

$$\|y^k(v^k) - y^k(u)\| \le L(\|u\| + \|v^k\|)\|v^k - u\|\,,$$

$$\|z^k(v^k) - z^k(u)\| \le L(\|u\| + \|v^k\|)\|v^k - u\|\,.$$

Now we have $y^k(v^k) \to y(u)$ and $z^k(v^k) \to z(u)$, which follows from the estimations

$$\begin{aligned}
\|y^k(v^k) - y(u)\| &\le \|y^k(v^k) - y^k(u)\| + \|y^k(u) - y(u)\| \\
&\le L(\|u\| + \|v^k\|)\|v^k - u\| + \|y^k(u) - y(u)\|\,, \\
\|z^k(v^k) - z(u)\| &\le \|z^k(v^k) - z^k(u)\| + \|z^k(u) - z(u)\| \\
&\le L(\|u\| + \|v^k\|)\|v^k - u\| + \|y^k(u) - y(u)\|\,.
\end{aligned}$$

For $k = 1,\, 2,\, \cdots$, let v^k be such that $w = \frac{2(1-\lambda)}{t_k\lambda}\,(v^k - u)$, i. e. $v^k = u + \frac{\lambda}{2(1-\lambda)}\, t_k\, w$ and hence, $v^k \to u$. For every k, we have

$$\begin{aligned}
g(x^0 + t_k v^k) - g(x^0) &= t_k g'(x^0)u + t_k g'(x^0)(v^k - u) + \frac{1}{2} t_k^2 z(u) + o(t_k^2) \\
&= t_k g'(x^0)u + \frac{1}{2\,(1-\lambda)}\, t_k^2\, \left((1-\lambda)\, z(u) + \frac{2\,(1-\lambda)}{t_k}\, g'(x^0)(v^k - u)\right) + o(t_k^2) \\
&= t_k g'(x^0)u + \frac{1}{2\,(1-\lambda)}\, t_k^2\, \left((1-\lambda)\, z(u) + \lambda\, g'(x^0)\left(\frac{2(1-\lambda)}{t_k\lambda}\,(v^k - u)\right)\right) + o(t_k^2) \\
&= t_k g'(x^0)u + \frac{1}{2\,(1-\lambda)}\, t_k^2\, \left((1-\lambda)\, z(u) + \lambda\, g'(x^0)w\right) + o(t_k^2) \\
&\in -K[g(x^0)] - \operatorname{int} K[g(x^0)] + o(t^2) \subset -\operatorname{int} K[g(x^0)]\,,
\end{aligned}$$

the last inclusion is satisfied for k large enough. We get from here (repeating partially the reasoning from the proof of $\mathbb{N}_p'$) that $g(x^0 + t_k v^k) \in -\operatorname{int} K$ for all sufficiently large k. Passing to a subsequence, we may assume that all $x^0 + t_k v^k$ are feasible. The above chain of equalities and inclusions can be repeated substituting g, $z(u)$ and $K[g(x^0)]$ respectively by f, $y(u)$ and C. We obtain in such a way $f(x^0 + t_k v^k) - f(x^0) \subset -\operatorname{int} C$, which contradicts to x^0 w-minimizer for (1). □

Now we establish second-order necessary and sufficient optimality conditions in dual form. We assume as usual that $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are $C^{1,1}$ functions and $C \subset \mathbb{R}^m$, $K \subset \mathbb{R}^p$ are closed convex cones. Then for $x^0 \in \mathbb{R}^n$ we put

$$\Delta(x^0) = \left\{(\xi,\, \eta) \in C' \times K' \mid (\xi, \eta) \neq 0,\ \langle \eta,\, g(x^0)\rangle = 0,\ \langle \xi,\, f'(x^0)\rangle + \langle \eta,\, g'(x^0)\rangle = 0\right\}\,.$$

Theorem 5 (Second-order conditions, Dual form) *Consider problem (1) with f and g being $C^{1,1}$ functions, and C and K closed convex cones with nonempty interiors.*

(Necessary Conditions) *Let x^0 be a w-minimizer for (1). Then for each $u \in \mathbb{R}^n$ the following two conditions hold:*

$$\mathbb{N}'_p : \qquad (f'(x^0)u,\, g'(x^0)u) \notin -(\operatorname{int} C \times \operatorname{int} K[g(x^0)])\,,$$

$$\mathbb{N}''_d : \qquad \textit{if } (f'(x^0)u,\, g'(x^0)u) \in -(C \times K[g(x^0)] \setminus \operatorname{int} C \times \operatorname{int} K[g(x^0)])$$
$$\textit{then } \forall\, (y^0, z^0) \in (f(x^0), g(x^0))''_u : \exists\, (\xi^0, \eta^0) \in \Delta(x^0) : \langle \xi^0,\, y^0\rangle + \langle \eta^0,\, z^0\rangle \geq 0\,.$$

(Sufficient Conditions) *Let x^0 be a feasible point for problem (1). Suppose that for each $u \in \mathbb{R}^n \setminus \{0\}$ one of the following two conditions is satisfied:*

$$\mathbb{S}'_p : \qquad (f'(x^0)u,\, g'(x^0)u) \notin -(C \times K[g(x^0)])\,,$$

$$\mathbb{S}''_d : \qquad (f'(x^0)u,\, g'(x^0)u) \in -(C \times K[g(x^0)] \setminus \operatorname{int} C \times \operatorname{int} K[g(x^0)])$$
$$\textit{and } \forall\, (y^0, z^0) \in (f(x^0), g(x^0))''_u : \exists\, (\xi^0, \eta^0) \in \Delta(x^0) : \langle \xi^0,\, y^0\rangle + \langle \eta^0,\, z^0\rangle > 0\,.$$

Then x^0 is a i-minimizer of order two for problem (1).

Proof. Necessary Conditions. The necessity of condition $\mathbb{N}'_p$ is proved in Theorem 4. To complete the proof we show that condition $\mathbb{N}''_p$ implies $\mathbb{N}''_d$. Thus, we assume that the convex sets $F_+ = \operatorname{conv}\{(y^0, z^0),\, \operatorname{im}(f'(x^0),\, g'(x^0))\}$ and $F_- = (\operatorname{int} C \times \operatorname{int} K[g(x^0)])$ do not intersect. Since $K \subset K[g(x^0)]$ and both C and K have nonempty interior, we see that both F_+ and F_- are nonempty. According to the Separation Theorem there exists a pair $(\xi^0, \eta^0) \neq (0,\, 0)$ and a real number α, such that

$$\begin{aligned} &\langle \xi^0,\, y\rangle + \langle \eta^0,\, z\rangle \leq \alpha \quad \text{for all} \quad (y, z) \in F_-\,, \\ &\langle \xi^0,\, y\rangle + \langle \eta^0,\, z\rangle \geq \alpha \quad \text{for all} \quad (y, z) \in F_+\,. \end{aligned} \tag{5}$$

Since F_- is a cone and F_+ contains the cone $\operatorname{im}(f'(x^0),\, g'(x^0))$, we get $\alpha = 0$. Since moreover, $\operatorname{im}(f'(x^0),\, g'(x^0))$ is a linear space, we see that $\langle \xi^0,\, y\rangle + \langle \eta^0,\, z\rangle = 0$ for all $(\xi, \eta) \in \operatorname{im}(f'(x^0), g'(x^0))$. The first line in in (5) gives now $(\xi^0, \eta^0) \in C' \times K[g(x^0)]'$, that is $(\xi^0, \eta^0) \in \Delta(x^0)$. With regard to $(y^0, z^0) \in F_+$, the second line gives $\langle \xi^0,\, y^0\rangle + \langle \eta^0,\, z^0\rangle \geq 0$.

Sufficient Conditions. We prove that if x^0 is not a i-minimizer of order two for (1), then there exists $u^0 \in \mathbb{R}^n$, $\|u^0\| = 1$, for which neither of the conditions $\mathbb{S}'_p$ and $\mathbb{S}''_d$ is satisfied.

Choose a monotone decreasing sequence $\varepsilon_k \to 0^+$. Since x^0 is not a i-minimizer of order two, there exist sequences $t_k \to 0^+$ and $u^k \in \mathbb{R}^n$, $\|u^k\| = 1$, such that $g(x^0 + t_k u^k) \in -K$ and

$$D(f(x^0 + t_k u^k) - f(x^0), -C) = \max_{\xi \in \Gamma_{C'}} \langle \xi,\, f(x^0 + t_k u^k) - f(x^0)\rangle < \varepsilon_k t_k^2\,. \tag{6}$$

Passing to a subsequence, we may assume $u^k \to u^0$. We may assume also $y^{0,k} \to y^0$, $z^{0,k} \to z^0$, where we have put

$$y^{0,k} = \frac{2}{t_k^2}\left(f(x^0 + t_k u^0) - f(x^0) - t_k\, f'(x^0)u^0\right)\,,$$

$$y^{k} = \frac{2}{t_k^2}\left(f(x^0 + t_k u^k) - f(x^0) - t_k\, f'(x^0)u^k\right)\,,$$

$$z^{0,k} = \frac{2}{t_k^2}\left(g(x^0 + t_k u^0) - g(x^0) - t_k\, g'(x^0)u^0\right)\,,$$

$$z^{k} = \frac{2}{t_k^2}\left(g(x^0 + t_k u^k) - g(x^0) - t_k\, g'(x^0)u^k\right)\,.$$

The possibility to choose convergent subsequence $y^{0,k} \to y^0$, $z^{0,k} \to z^0$, follows from the boundedness of the differential quotient proved in Lemma 1. Now we have $(y^0, z^0) \in (f(x^0), g(x^0))''_u$. We may assume also that $0 < t_k < r$ and both f and g are Lipschitz with constant L on $\{x \mid \|x - x^0\| \le r\}$.
Now we have $z^k \to z^0$ which follows from the estimations obtained on the base of Lemma 1

$$\|z^k - z^0\| \le \|z^k - z^{0,k}\| + \|z^{0,k} - z^0\|$$

$$\le L(\|u^k\| + \|u^0\|)\, \|u^k - u^0\| + \|z^{0,k} - z^0\|\,.$$

Similar estimations for y^k show that $y^k \to y^0$.

We prove that $\mathbb{S}'_p$ is not satisfied at u^0. For this purpose we must prove that

$$f'(x^0)u \in -C\,, \quad g'(x^0)u \in -K[g(x^0)]\,.$$

Let $\varepsilon > 0$. We claim that there exists k_0 such that for all $\xi \in \Gamma_{C'}$ and all $k > k_0$ the following inequalities hold:

$$\langle \xi,\, \frac{1}{t_k}\Big(f(x^0 + t_k u^k) - f(x^0)\Big)\rangle < \frac{1}{3}\,\varepsilon\,, \tag{7}$$

$$\langle \xi,\, f'(x^0)u^k - \frac{1}{t_k}\,\Big(f(x^0 + t_k u^k) - f(x^0)\Big)\rangle < \frac{1}{3}\,\varepsilon\,, \tag{8}$$

$$\langle \xi,\, f'(x^0)(u^0 - u^k)\rangle < \frac{1}{3}\,\varepsilon\,. \tag{9}$$

Inequality (7) follows from (6). Inequality (8) follows from the Fréchet differentiability of f

$$\langle \xi,\, f'(x^0)u^k - \frac{1}{t_k}\,\Big(f(x^0 + t_k u^k) - f(x^0)\Big)\rangle$$

$$\le \|\frac{1}{t_k}\,\Big(f(x^0 + t_k u^k) - f(x^0)\Big) - f'(x^0)u^k\| < \frac{1}{3}\,\varepsilon\,,$$

which is true for all sufficiently small t_k. Inequality (9) follows from

$$\langle \xi,\, f'(x^0)(u^0 - u^k)\rangle \le \|f'(x^0)\|\, \|u^0 - u^k\| < \frac{1}{3}\,\varepsilon\,,$$

which is true for $\|u^k - u^0\|$ "small enough". Now we see, that for arbitrary $\xi \in \Gamma_{C'}$ and $k > k_0$ we have

$$\begin{aligned}\langle \xi,\, f'(x^0)u^0\rangle = \langle \xi,\, \frac{1}{t_k}\Big(f(x^0 + t_k u^k) - f(x^0)\Big)\rangle \\ +\langle \xi,\, f'(x^0)u^k - \frac{1}{t_k}\Big(f(x^0 + t_k u^k) - f(x^0)\Big)\rangle \\ +\langle \xi,\, f'(x^0)(u^0 - u^k)\rangle < \frac{1}{3}\,\varepsilon + \frac{1}{3}\,\varepsilon + \frac{1}{3}\,\varepsilon = \varepsilon\,,\end{aligned}$$

and $D(f'(x^0)u^0, -C) = \max_{\xi \in \Gamma}\langle \xi,\, f'(x^0)u^0\rangle < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we see $D(f'(x^0)u^0, -C) \le 0$, whence $f'(x^0)u^0 \in -C$.
Similar estimations can be repeated with f and $\xi \in \Gamma_{C'}$ substituted respectively by g and $\eta \in K[g(x^0)]'$, whence we get $g'(x^0)u^0 \in -K[g(x^0)]$. The only difference occurs with the estimation (7). Then we have

$$\langle \eta,\, \frac{1}{t_k}\Big(g(x^0 + t_k u^k) - g(x^0)\Big)\rangle = \frac{1}{t_k}\,\langle \eta,\, g(x^0 + t_k u^k)\rangle \le 0\,,$$

since $g(x^0 + t_k u^k) \in -K \subset -K[g(x^0)]$.

Now we prove that $\mathbb{S}_d''$ is not satisfied at u^0. We assume that

$$(f'(x^0)u,\, g'(x^0)u) \in -(C \times K[g(x^0)] \setminus \operatorname{int} C \times \operatorname{int} K[g(x^0)])\,,$$

since otherwise the first assertion in condition $\mathbb{S}_d''$ would not be satisfied. It is obvious, that to complete the proof we must show that

$$\forall\,(\xi^0,\eta^0) \in \Delta(x^0) : \langle \xi^0,\, y^0\rangle + \langle \eta^0,\, z^0\rangle \le 0\,.$$

Obviously, it is enough to show this inequality for $(\xi^0,\eta^0) \in \Delta(x^0)$ such that $\max\big(\|\xi^0\|,\, \|\eta^0\|\big) = 1$. Then

$$\langle \xi^0,\, y^0\rangle + \langle \eta^0,\, z^0\rangle = \lim_k \Big(\langle \xi^0,\, y^k\rangle + \langle \eta^0,\, z^k\rangle\Big)$$

$$= \lim_k \bigg(\frac{2}{t_k^2}\,\langle \xi^0,\, f(x^0+t_k u^k) - f(x^0)\rangle + \frac{2}{t_k^2}\,\langle \eta^0,\, g(x^0+t_k u^k) - g(x^0)\rangle$$

$$- \frac{2}{t_k^2}\,\Big(\langle \xi^0,\, f'(x^0)u^k\rangle + \langle \eta^0,\, g'(x^0)u^k\rangle \le 0\Big)\bigg)$$

$$\le \limsup_k \frac{2}{t_k^2}\, D(f(x^0+t_k u^k) - f(x^0), -C) + \limsup_k \frac{2}{t_k^2}\,\langle \eta^0,\, g(x^0+t_k u^k)\rangle$$

$$\le \limsup_k \frac{2}{t_k^2}\,\varepsilon_k t_k^2 = 0\,.$$

□

In fact, in Theorem 5 only conditions $\mathbb{N}_d''$ and $\mathbb{S}_d''$ are given in a dual form. Because of convenience for the proof conditions $\mathbb{N}_p'$ and $\mathbb{S}_p'$ are given in the same primal form as they appear in Theorem 4. However these conditions can be represented in a dual form too. For instance condition $\mathbb{N}_p'$ can be written in the equivalent dual form

$$\exists\,(\xi,\,\eta) \in (C' \times K[g(x^0)]') : \langle \xi,\, f'(x^0)u\rangle + \langle \eta,\, g'(x^0)u\rangle > 0\,.$$

Theorems 3.1 and 4.2 in Liu, Neittaanmäki, Křížek [20] are of the same type as Theorem 5. The latter is however more general in the following aspects. Theorem 5 in opposite to [20] concerns arbitrary and not only polyhedral cones C. In Theorem 5 the conclusion in the sufficient conditions part is that x^0 is a i-minimizer of order two, while in [20] the weaker conclusion is proved, namely that the reference point is only e-minimizer.

5 Reversal of the sufficient conditions

The isolated minimizers as more qualified notion of efficiency should be more essential solutions for the vector optimization problem than the e-minimizers. As a result of using i-minimizers, we have seen in Theorem 1 that under suitable constraint qualifications the sufficient conditions admit reversal. Now we discuss the similar possibility to revert the sufficient conditions part of Theorem 5 under suitable second-order constraint qualifications of Kuhn-Tucker type. Unfortunately, the second-order case is more complicated than the first-order one. The proposed second-order constraint qualifications called $\mathbb{Q}_{1,1}(x^0)$ look more complicated than the first-order ones applied in Theorem 1. In spite of this, we succeeded in proving that conditions from the sufficient part of Theorem 5 are satisfied under the stronger hypothesis that x^0 is a i-minimizer of order two for the following constrained problem

$$\min\nolimits_{\hat{C}} f(x), \quad g(x) \in -K, \quad \text{where} \quad \hat{C} = \operatorname{conv}\,\big(C,\, \operatorname{im} f'(x^0)\big)\;. \tag{10}$$

Obviously, each i-minimizer of order two for (10) is also a i-minimizer of order two for (1). It remains the open question, whether under the hypotheses that x^0 is a i-minimizer of order two for (1) the conclusion of Theorem 6 remains true.

In the sequel we consider the constraint qualification.

$\mathbb{Q}_{1,1}(x^0)$: The following conditions are satisfied:

1^0. $$g(x^0) \in -K\,,$$

2^0. $$g'(x^0)u^0 \in -(K[g(x^0)] \setminus \operatorname{int} K[g(x^0)])\,,$$

3^0. $$\frac{2}{t_k^2}\,\left(g(x^0+t_k u^0) - g(x^0) - t_k g'(x^0)u^0\right) \to z^0\,,$$

4^0. $(\forall\, \eta \in K[g(x^0)]', (\forall\, z \in \operatorname{im} g'(x^0) : \langle \eta,\, z\rangle = 0) : \langle \eta,\, z^0\rangle \le 0)$ implies
$\exists\, u^k \to u^0 : \exists\, k_0 \in \mathbb{N} : \forall\, k > k_0 : g(x^0+t_k u^k) \in -K$.

Theorem 6 (Reversal of the second-order sufficient conditions) *Let x^0 be a i-minimizer of order two for the constrained problem (10) and suppose that the constraint qualification $\mathbb{Q}_{1,1}(x^0)$ holds. Then one of the conditions $\mathbb{S}'_p$ or $\mathbb{S}''_d$ from Theorem 5 is satisfied.*

Proof. Let x^0 be a i-minimizer of order two for problem (10), which means that $g(x^0) \in -K$ and there exists $r > 0$ and $A > 0$ such that $g(x) \in -K$ and $\|x - x^0\| \le r$ implies

$$D(f(x) - f(x^0), -\hat{C}) = \max_{\xi \in \Gamma_{\hat{C}'}} \langle \xi, f(x) - f(x^0)\rangle \ge A\,\|x - x^0\|^2\,. \tag{11}$$

The point x^0 being a i-minimizer of order two for problem (10) is also a w-minimzer for (10) and hence also for (1). Therefore it satisfies condition $\mathbb{N}'_p$. Now it becomes obvious, that for each $u = u^0$ one and only one of the first-order conditions in $\mathbb{S}'_p$ and the first part of $\mathbb{S}''_d$ is satisfied. Suppose that $\mathbb{S}'_p$ is not satisfied. Then the first part of condition $\mathbb{S}''_d$ holds, that is

$$(f'(x^0)u^0,\, g'(x^0)u^0) \in -(C \times K[g(x^0)] \setminus \operatorname{int} C \times \operatorname{int} K[g(x^0)])\,.$$

We prove, that also the second part of condition $\mathbb{S}''_d$ holds.

One of the following two cases may arise:

1^0. *There exists $\eta^0 \in K[g(x^0)]'$ such that $\langle \eta^0,\, z\rangle = 0$ for all $z \in \operatorname{im} f'(x^0)$ and $\langle \eta^0,\, z^0\rangle > 0$.*

We put now $\xi^0 = 0$. Then we have obviously $(\xi^0, \eta^0) \in C' \times K[g(x^0)]'$ and $\langle \xi^0,\, f'(x^0)\rangle + \langle \eta^0,\, g'(x^0)\rangle = 0$. Thus $(\xi^0, \eta^0) \in \Delta(x^0)$ and $\langle \xi^0,\, y^0\rangle + \langle \eta^0,\, z^0\rangle > 0$. Therefore condition $\mathbb{S}''_d$ is satisfied.

2^0. *For all $\eta \in K[g(x^0)]'$, such that $\langle \eta,\, z\rangle = 0$ for all $z \in \operatorname{im} f'(x^0)$, it holds $\langle \eta,\, z^0\rangle \le 0$.*

This condition coincides with condition 4^0 in the constraint qualification $\mathbb{Q}_{1,1}(x^0)$. Now we see that all points 1^0–4^0 in the constraint qualification $\mathbb{Q}_{1,1}(x^0)$ are satisfied. Therefore there exists $u^k \to u^0$ and a positive integer k_0 such that for all $k > k_0$ it holds $g(x^0 + t_k u^k) \in -K$. Passing to a subsequence, we may assume that this inclusion holds for all k. From (11) it follows that there exists $\xi^k \in \Gamma_{\hat{C}'}$ such that

$$D(f(x^0+t_k u^k) - f(x^0), -\hat{C}) = \langle \xi^k,\, f(x^0+t_k u^k) - f(x^0)\rangle \ge A\,t_k^2 \|u^k\|^2\,.$$

Passing to a subsequence, we may assume that $\xi^k \to \xi^0 \in \Gamma_{\hat{C}'}$. Let us underline that each $\xi \in \hat{C}'$ satisfies $\langle \xi,\, f'(x^0)\rangle = 0$. This follows from $\langle \xi,\, z\rangle \le 0$ for all $z \in \operatorname{im} f'(x^0)$ and the fact that $\operatorname{im} f'(x^0)$ is a linear space. Now we have

$$\langle \xi^k,\, \frac{2}{t_k^2}\,\Big(f(x^0+t_k u^k) - f(x^0) - t_k f'(x^0)u^k\Big)\rangle$$
$$\frac{2}{t_k^2}\,\langle \xi^k,\, \Big(f(x^0+t_k u^k) - f(x^0)\Big)\rangle \ge \frac{2}{t_k^2}\,A\,t_k^2\,\|u^k\|^2 = A\,\|u^k\|^2\,.$$

Passing to a limit gives $\langle \xi^0, y^0 \rangle \geq 2\,A\,\|u^0\|^2 = 2\,A$. Putting $\eta^0 = 0$, we have obviously $(\xi^0, \eta^0) \in \Delta(x^0)$ and $\langle \xi^0, y^0 \rangle + \langle \eta^0, z^0 \rangle > 0$. $\square$

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