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First order optimality conditions in set-valued optimization

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Abstract

A a set-valued optimization problem $\min_C F(x)$, $x \in X_0$, is considered, where $X_0 \subset X$, X and Y are Banach spaces, $F : X_0 \rightsquigarrow Y$ is a set-valued function and $C \subset Y$ is a closed cone. The solutions of the set-valued problem are defined as pairs (x^0, y^0), $y^0 \in F(x^0)$, and are called minimizers. In particular the notions of w-minimizer (weakly efficient points), p-minimizer (properly efficient points) and i-minimizer (isolated minimizers) are introduced and their characterization in terms of the so called oriented distance is given. The relation between p-minimizers and i-minimizers under Lipschitz type conditions is investigated. The main purpose of the paper is to derive first order conditions, that is conditions in terms of suitable first order derivatives of F, for a pair (x^0, y^0), where $x^0 \in X_0$, $y^0 \in F(x^0)$, to be a solution of this problem. We define and apply for this purpose the directional Dini derivative. Necessary conditions and sufficient conditions a pair $(x^0,\, y^0)$ to be a w-minimizer, and similarly to be a i-minimizer are obtained. The role of the i-minimizers, which seems to be a new concept in set-valued optimization, is underlined. For the case of w-minimizers some comparison with existing results is done.

Key words: Vector optimization, Set-valued optimization, First-order optimality conditions.
Math. Subject Classification: 90C29, 90C30, 90C46, 49J52.

1 Introduction

We consider the set-valued optimization problem

$$\min_C F(x)\,, \quad x \in X_0\,, \tag{1}$$

where $X_0 \subset X$, X and Y are Banach spaces, $F : X_0 \rightsquigarrow Y$ is a set-valued function (following [1] we use the squiggled arrow $\rightsquigarrow$ to denote a set-valued function and the usual arrow $\rightarrow$ for a single-valued one). We suppose that $C \subset Y$ is a closed cone. We confine usually to convex cones, but occasionally we underline that some of the results are true also for nonconvex cones. Let us make the remark, that though a nonconvex cone C does not introduce a partial order in Y, the given further definitions of optimality formally have sense also for nonconvex cones. The aim of the paper is to derive first order conditions, that is conditions in terms of suitable first order derivatives of F, a pair (x^0, y^0), where $x^0 \in X_0$, $y^0 \in F(x^0)$, to be a solution of this problem. We apply the directional Dini derivatives defined in Section 3 in feasible directions. Recall that a feasible direction for X_0 at x^0 is any $u \in X$ such that $x^0 + tu \in X_0$ for all t, $0 < t < \delta$, and some $\delta > 0$. The set of the feasible directions for X_0 at x^0 is denoted $X_0(x^0)$. Obviously $X_0(x^0)$ is a cone in X. Of some importance in our considerations are the points $x^0 \in X_0$ at which X^0 is locally convex. We say that X_0 is locally convex at x^0 if there exists $\gamma > 0$ such that $X_0 \cap (x^0 + \gamma \operatorname{cl} B_X) = \{x \in X_0 \mid \|x - x^0\| \le \gamma\}$ is convex. Obviously, then $X_0 \cap (x^0 + \gamma \operatorname{cl} B_X) \subset x^0 + X_0(x^0)$, that is for any $x \in X_0 \cap \gamma \operatorname{cl} B_X$ it holds $x - x^0 \in X_0(x^0)$. This condition is satisfied if $x^0 \in \operatorname{int} X_0$. It is also satisfied if X_0 is a convex set and $x^0 \in X_0$ is arbitrary. In Section 5 we give a definition of a star shaped at x^0 set. If X_0 is star shaped at x^0 then it need not be locally convex, but $X_0 \subset x^0 + X_0(x^0)$, that is each direction $x - x^0$, $x \in X_0$, is feasible.

For the considered closed convex cone C we suppose usually that $C \neq Y$, but whenever assumed, this will be written explicitly.

Saying that a set-valued function $F : X_0 \rightsquigarrow Y$ is given, we suppose that $F(x) \neq \emptyset$ for $x \in X_0$. We do not fix in advance other assumptions for F, gaining in such a way the freedom to analyze and associate to each artifact the appropriate hypotheses. The main idea of the paper is to generalize from vector-valued to set-valued optimization the results from [9]. A vector-valued problem, see below (3), occurs when a single-valued (we use as synonym vector-valued) function f instead of the set-valued function F is considered. The concept of an isolated minimizer is put into the center of the investigations in [9] and the obtained there results concern finite-dimensional spaces, an assumption which is important also for the present paper. In fact some of the results concern the case of finite dimensional image space Y. In this paper, in particular, we generalize the concept of an isolated minimizer to the set-valued problem (1). In the sequel we will use the abbreviations: svf for set-valued function, svp for set-valued problem, vvf for vector-valued function, vvp for vector-valued problem.

The solutions of svp (1) are defined as pairs (x^0, y^0), $y^0 \in F(x^0)$. With exception of Section 5 we consider solutions in a local sense. Similarities with vector optimization problems allow to use in set-valued optimization notions from vector optimization. In particular, the solutions of svp (1) can be called efficient points. We prefer, like in scalar optimization, to call them minimizers. In Section 2 we define different type of minimizers and give their characterizations in terms of the so called oriented distance. Among them the notions of w-minimizer (weakly efficient point), p-minimizer (properly efficient point) and i-minimizer (isolated minimizer) play an important role. The concept of a C-Lipschitz svf is introduced. It is shown that in the case when F is C-Lipschitz each i-minimizer is a p-minimizer. In Section 3 we give necessary conditions a pair (x^0, y^0) to be w-minimizer and sufficient conditions it to be i-minimizer for (1). The reversal in case of an i-minimizer is also obtained. Section 4 in case of C-Lipschitz data establishes conditions under which a p minimizer is a i-minimizer. Section 5 discusses the reversal of the necessary conditions for w-minimizers and establishes such a possibility under convexity type conditions. In Section 6 we compare the obtained results with those of [19].

Recently the interest toward set-valued optimization has grown up. To some extend the impetus is due to the advances in non-smooth and set-valued analysis. The concept of epiderivative dominates however in most of the investigations. Optimality crireria based on Clarke epiderivative are introduced in [24]. Contingent epiderivatives for the purpose of set-valued optimization are defined in [19] and thereafter developed in [17] and [18]. The contingent epiderivatives are generalized in [4], [5], [7]. Fenchel type duality in set-valued optimization appears in [27]. Let us underline that the epiderivatives are dual notions and often are used to treat set-valued optimality under convexity [16], [19], [27]. In our opinion primal concepts as directional derivatives give more freedom to be applied in more general situations (see further Remarks 3.1 and 3.2). As an illustration of this claim in Section 6 we make comparison of some of the results of the present paper and those of [19]. An attempt to treat set-valued optimization through directional derivatives is undertaken also in [23], where a necessary condition for proper efficiency is proved. Since the result in [23] is based on a different approach and formulated in different terms than the presented here, we prefer to postpone the discussion on the eventual comparison of our results with that of [23].

2 Concepts of optimality

Here $\mathbb{R}$ is the set of the reals and $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{-\infty\}$ its two point extension with the infinite elements. We put also $\mathbb{R}_+ = [0, +\infty)$ and $\mathbb{R}_- = (-\infty, 0]$. For the the norm and the dual pairing in the considered normed spaces X and Y we write $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$. From the context it should be clear to exactly which spaces these notations are applied. We denote by $B_X = \{x \in X \mid \|x\| < 1\}$ and $B_Y = \{y \in Y \mid \|y\| < 1\}$ the open unit balls respectively in X and Y. Similarly, the notations $S_X = \{x \in X \mid \|x\| = 1\}$ and $S_Y = \{y \in Y \mid \|y\| = 1\}$ are used for the unit spheres. In the cases when X or Y are finite dimensional, of dimension n and m respectively, we will identify them with the Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, taking advantage of the particular properties of Euclidean metrics (see below Lemma 2.2). Let us underline, that because of the invariance of the defined below concept of the optimality with respect to equivalent norms, the results proved for Euclidean spaces remain true also for finite dimensional Banach spaces.

The notion of the positive polar cone is used in the sequel. We recall that for the given closed convex cone $C \subset Y$ its positive polar cone is defined by $C' = \{\xi \in Y \mid \langle \xi, y \rangle \geq 0 \text{ for all } y \in C\}$. For a set $A \subset$ and $y^0 \in Y$ we will make use of $\operatorname{cone}(A - y^0) := \{\lambda(y - y^0) \mid \lambda \geq 0,\, y \in A\}$.

We introduce the following concepts of solutions for problem (1). The pair (x^0, y^0), $y^0 \in F(x^0)$, is said to be w-minimizer (respectively e-minimizer) if there exists a neighbourhood U of x^0 such that if $x \in U \cap X_0$ then $F(x) \cap \big(y^0 - \operatorname{int} C\big) = \emptyset$ (respectively $F(x) \cap \big(y^0 - (C \setminus \{0\}\big) = \emptyset$). In vector optimization w-minimizers are called weakly efficient points and e-minimizers efficient points. Obviously, if $C \neq Y$, each e-minimizer is w-minimizer.

Define now the weakly efficient frontier (w-frontier) $w\text{-Min}_C A$ and efficient frontier (e-frontier) $e\text{-Min}_C A$ of a set $A \subset Y$ with respect to the cone C by $w\text{-Min}_C A = \{y \in A \mid A \cap (y - \operatorname{int} C) = \emptyset\}$ and $e\text{-Min}_C A = \{y \in A \mid A \cap (y - (C \setminus \{0\})) = \emptyset\}$. If $C \neq Y$ it holds $\operatorname{int} C \subset C \setminus \{0\}$, whence $w\text{-Min}_C A \supset e\text{-Min}_C A$. For vector optimization theory based on notions of efficient frontiers see Luc [25].

Putting $x = x^0$ in the above definitions we see that if (x^0, y^0) is a w-minimizer (respectively e-minimizer) for svp (1) then y^0 belongs to the w-frontier (respectively e-frontier) of the set $F(x^0)$. Thus, in order that (x^0, y^0), $y^0 \in F(x^0)$, be a minimizer of certain type for svp (1) necessary some frontier-type limitations for the point y^0 do occur.

For a set $A \subset Y$ the distance from $y \in Y$ to A is given by $d(y, A) = \inf\{\|a - y\| \mid a \in A\}$. It is convenient to allow also value $+\infty$ of the distant function putting $d(y, \emptyset) = +\infty$.

The oriented distance from y to A is defined by $D(y, A) = d(y, A) - d(y, Y \setminus A)$. It takes values in $\overline{\mathbb{R}}$ and in particular $D(y, \emptyset) = +\infty$ and $D(y, Y) = -\infty$. The function D is introduced in Hiriart-Urruty [14], [15], and since then is often used in vector optimization. Ginchev, Hoffmann [12] apply the oriented distance to study approximation of set-valued functions by single-valued ones and in case of a convex set A show the representation $D(y, A) = \sup_{\|\xi\|=1} \left(\langle \xi, y \rangle - \sup_{a \in A} \langle \xi, a \rangle\right)$. Let us underline that this formula works also for $A = \emptyset$ or $A = Y$. From this representation, if $A = -C$ is a convex cone and taking into account

$$\sup_{a \in -C} \langle \xi, a \rangle = \begin{cases} 0 & , \quad \xi \in C', \\ +\infty & , \quad \xi \notin C', \end{cases}$$

we get easily $D(y, -C) = \sup_{\|\xi\|=1,\, \xi \in C'} \left(\langle \xi, y \rangle\right)$.

We define next the oriented distance $D(M, A)$ from a set $M \subset Y$ to the set $A \subset Y$ putting $D(M, A) = \inf\{D(y, A) \mid y \in M\}$.

A characterization of w-minimizers can be obtained in terms of the oriented distance.

Proposition 2.1 *The pair (x^0, y^0), $y^0 \in F(x^0)$, is a w-minimizer of svp (1) with $C \neq Y$ if and only if $\varphi(x^0) = 0$ and x^0 is a minimizer for the scalar function*

$$\varphi : X_0 \to \mathbb{R}\,, \quad \varphi(x) = D(F(x) - y^0, -C)\,. \tag{2}$$

Proof Let (x^0, y^0) be a w-minimizer. Then there is a neighbourhood U of x^0 such that $F(x) \cap \big(y^0 - \operatorname{int} C\big) = \emptyset$ for $x \in U \cap X_0$, whence $\varphi(x) = D(F(x) - y^0, -C) \geq 0$. In particular $\varphi(x^0) \geq 0$. On the other hand $y^0 \in F(x^0)$ gives $\varphi(x^0) \leq D(0, -C) = 0$. Thus $\varphi(x^0) = 0$ and x^0 is a minimizer of the scalar function (2). Conversely, let x^0 be a minimizer of φ and $\varphi(x^0) = 0$. Then $\varphi(x) \geq \varphi(x^0) = 0$ for $x \in U \cap X_0$. This inequality shows that $F(x) \cap \big(y^0 - \operatorname{int} C\big) = \emptyset$. $\square$

If instead of svf F we consider vvf $f : X_0 \to Y$, we come to the vvp

$$\min_C f(x)\,, \quad x \in X_0\,. \tag{3}$$

For this problem the function (2) is

$$\varphi : X_0 \to \mathbb{R}\,, \quad \varphi(x) = D(f(x) - f(x^0), -C)\,. \tag{4}$$

Proposition 2.1 gives now that x^0 is w-minimizer of vvp (3) if and only if x^0 is a minimizer for the scalar function (4). In this case the condition $\varphi(x^0) = 0$ from the assumptions of Proposition 2.1 is superfluous, since it is automatically satisfied as a consequence of the single-valuedness of f.

For vvp (3) the following concept of optimality looks natural. We call the point x^0 a strong minimizer (s-minimizer) of vvp (3) if the point x^0 is a strong minimizer for the scalar function (4), the latter means $\varphi(x) > \varphi(x^0)$ for $x \in (U \setminus \{x^0\}) \cap X_0$. Equivalently, x^0 is a s-minimizer for vvp (3) if $f(x) - f(x^0) \notin -C$ for $x \in (U \setminus \{x^0\}) \cap X_0$. Obviously, for vvp (3) each s-minimizer is e-minimizer. Now, if we wish to define the notion of a s-minimizer for svp (1) in a way that it maintains this relation between s-minimizers and e-minimizers, we have to keep attention, that the property (x^0, y^0) e-minimizer for svp (1) implies that y^0 belongs to the efficient boundary $e\text{-}Min_C F(x^0)$ of $F(x^0)$ and the latter must be explicitly said in the definition of a s-minimizer of (1).

With regard to the above remark, we come easily to the following definition. We call (x^0, y^0), $y^0 \in F(x^0)$, a strong minimizer (s-minimizer) for svp (1), if there is a neighbourhood U of x^0 such that $F(x) \cap (y^0 - C) = \emptyset$ for $x \in (U \setminus \{x^0\}) \cap X_0$ and $F(x^0) \cap (y^0 - C) = \{y^0\}$. From this definition it follows that if x^0 is a s-minimizer for svp (1) then $x^0 \in e\text{-}\mathrm{Min}_C F(x^0)$.

Now as a consequence of Proposition 2.1 we get obviously the following characterization of the s-minimizers of svp (1).

Proposition 2.2 *The pair (x^0, y^0), $y^0 \in F(x^0)$, is a s-minimizer for svp (1) with $C \neq Y$ if and only if x^0 is a strong minimizer for the scalar function (2) and $F(x^0) \cap (y^0 - C) = \{y^0\}$.*

Next we recall the notion of a properly efficient point (p-minimizer) for vvp (3). In the case when C is a pointed convex cone the following definition is well known [13]. The point x^0 is said to be a p-minimizer for vvp (3) if there exists a closed convex cone $\tilde{C} \subset \mathbb{R}^m$ satisfying $C \setminus \{0\} \subset \operatorname{int} \tilde{C}$, such that x^0 is a w-minimizer for the problem $\min_{\tilde{C}} f(x)$, $x \in X_0$. However, we consider the assumption C pointed as not natural for eventual extension of the theory toward constrained problems (compare with [9] and [11]). For this reason we prefer the following definition.

Let $C \subset \mathbb{R}^m$ be a cone and let a be a real number. Define the set

$$C(a) = \{y \in Y \mid y \in Y \mid D(y,\, C) \le a\, \|y\|\}\,.$$

The set $C(a)$ is a closed (but not necessarily convex) cone, which is a consequence of the positive homogeneity of the oriented distance $D(\cdot, C)$ and the norm $\|\cdot\|$.

We say [11] that x^0 is a p-minimizer for vvp (3) if there exists a, $0 < a < 1$, and a neighbourhood U of x^0 such that $f(x) - f(x^0) \notin -\operatorname{int} C(a)$ for $x \in U \cap X_0$.

Let us turn attention, that when C is pointed closed convex cone, Y is finite dimensional, and $a > 0$ is sufficiently small, then $C(a)$ is also a pointed closed convex cone, whence our definition of a p-minimizer coincides with the commonly accepted one.

Similarly, we say that the point (x^0, y^0), $y^0 \in F(x^0)$, is a p-minimizer for svp (1) if there exists a, $0 < a < 1$, and a neighbourhood U of x^0, such that $x \in U \cap X_0$ and $y \in F(x)$ imply $y - y^0 \notin -\operatorname{int} C(a)$.

Given a set $A \subset Y$ we define the properly efficient frontier (p-frontier) of A with respect to C by

$$p\text{-}\mathrm{Min}_C A = \{y \in A \mid A \cap (y - C(a)) = \{y\} \text{ for some } a,\ 0 < a < 1\}\,.$$

Obviously $e\text{-}\mathrm{Min}_C A \supset p\text{-}\mathrm{Min}_C A$.

For $x = x^0$ the definition of a p-minimizer for svp (1) gives now that if (x^0, y^0), $y^0 \in F(x^0)$, is a p-minimizer for svp (1) then $y^0 \in p\text{-}\mathrm{Min}_C F(x^0)$.

Another concept of optimality is the concept of an isolated minimizer (i-minimizer). We say that (x^0, y^0), $y^0 \in F(x^0)$, is a i-minimizer for svp (1) if there is a neighbourhood U of x^0 and a constant $A > 0$ such that $D(F(x) - y^0, -C) \ge A\, \|x - x^0\|$ and $y^0 \in p\text{-}\mathrm{Min}_C F(x^0)$ for $x \in U \cap X_0$.

Generally, if $\varphi : X_0 \to \mathbb{R}$ is any scalar function, the point $x^0 \in X_0$ is said to be an isolated minimizer of order $\kappa > 0$ for φ, if there exists a neighbourhood U of x^0 and a constant $A > 0$, such that $\varphi(x) - \varphi(x^0) \ge A\, \|x - x^0\|^\kappa$ for $x \in U \cap X_0$. In this paper we deal only with isolated minimizers of order 1. The notion of isolated minimizer has been popularized by Auslender [2]. For vector functions it has been extended by Ginchev [8], Ginchev, Guerraggio, Rocca [9], [10], [11] and under the name of strict efficiency by Jiménez [20], [21], and Jiménez, Novo [22]. We prefer to use the original name of *isolated minimizer* given by Auslender. Besides, the concept of a strict minimizer has been used in vector optimization also in another meaning, see e. g. [3] and [6].

In the definition of a i-minimizer for svp appears explicitly the inclusion $y^0 \in p\text{-}\mathrm{Min}_C F(x^0)$. Now we give some explanation. For vvp (3) with locally Lipschitz function f each i-minimizer is also a p-minimizer, see [6], [9]. In order that similar relation occurs for svp (1), see below Theorem 2.1, we need to insert explicitly this assumption. It is necessary satisfied for a p-minimizer and does not follow from inequality $D(F(x) - y^0, -C) \ge A\, \|x - x^0\|$ being used in the definition of a i-minimizer for svp (1).

Obviously, the following characterization of the i-minimizers holds.

Proposition 2.3 *The pair (x^0, y^0), $y^0 \in F(x^0)$, is a i-minimizer for svp (1) if and only if x^0 is an isolated minimizer of first order for the scalar function (2) and $F(x^0) \cap (y^0 - C(a)) = \{y^0\}$ for some a, $0 < a < 1$.*

We recall [1] that the svf $F : X_0 \rightsquigarrow Y$ is locally Lipschitz at $x^0 \in X_0$, if there exists a neighbourhood U of x^0 and a constant $L > 0$, such that for $x^1,\, x^2 \in U \cap X_0$ it holds $F(x^2) \subset F(x^1) + L\, \|x^2 - x^1\|\, B_Y$. The svf $F : X_0 \rightsquigarrow Y$ is locally Lipschitz, if it is locally Lipschitz at each $x^0 \in X_0$.

In the paper we consider also set-valued functions obeying locally Lipschitz property with respect to the given closed convex cone C. We say that svf $F : X_0 \rightsquigarrow Y$ is locally Lipschitz w.r.t. C at $x^0 \in X_0$, or locally C-Lipschitz at x^0, if there exists a neighbourhood U of x^0 and a constant $L > 0$ such that it holds

$$F(x^2) \subset F(x^1) + C + L\,\|x^2 - x^1\|\,\mathrm{cl}\,B_Y \quad \text{for all} \quad x^1, x^2 \in U \cap X_0\,. \tag{5}$$

We say that svf $F : X_0 \rightsquigarrow Y$ is locally C-Lipscitz if it is locally Lipschitz at each $x^0 \in X_0$. Let us underline, that because of the convexity of C, svf F is locally C-Lipschitz if and only if the set-valued function $x \rightsquigarrow F(x) + C$ is locally Lipschitz.

In Theorem 2.1 we give a relation between i-minimizers and p-minimizers in the case when the image space Y is finite dimensional. As we underlined then Y can be identified with the Euclidean space $\mathbb{R}^n$. It seems that this relation remains true also in case of Y infinite dimensional. In fact, it is true for Y Hilbert space, since then the proof of the following Lemmas 2.1 and 2.2 is nearly the same. For the case of Y arbitrary Banach obviously the following lemmas need some modification.

Lemma 2.1 *Let $C \subset \mathbb{R}^m$ be a closed (not necessarily convex) cone and $a_1,\, a_2 > 0$ be two positive numbers. Then $C(a_1)(a_2) \subset C(a_1 + a_2)$.*

Proof Let $y \in C(a_1)(a_2)$. We must show that $y \in C(a_1 + a_2)$. The case $y \in C(a_1)$ is obvious. Suppose now $y \notin C(a_1)$. Let y' be a orthogonal projection of y on the cone $C(a_1)$ and $d' = D(y, C(a_1)) = \|y - y'\|$. The definition of $C(a_1)(a_2)$ yields $d' \le a_2\|y\|$ and from the properties of the orthogonal projection we have $\|y'\| \le \|y\|$ (in fact $\|y\|^2 = \|y'\|^2 + \|y - y'\|^2$, true for an Euclidean space). Denote by $\bar y$ and $\bar y'$ orthogonal projections respectively of y and y' on C. Put $\bar d = D(y, C) = \|y - \bar y\|$ and $\bar d' = D(y', C) = \|y' - \bar y'\|$. Then obviously it holds

$$\bar d = \|y - \bar y\| \le \|y - \bar y'\| \le \|y - y'\| + \|y' - \bar y'\|$$

$$\le a_2\,\|y\| + a_1\,\|y'\| \le (a_1 + a_2)\,\|y\|\,.$$

Therefore $y \in C(a_1 + a_2)$. □

Lemma 2.2 *Let in svp (1) with $Y = \mathbb{R}^m$ and closed (not necessarily convex) cone C the svf $F : X_0 \rightsquigarrow Y$ be C-Lipschitz with constant L in the neighbourhood U of x^0 and $y^0 \in F(x^0)$. Suppose that for some $\sigma > 0$ it holds $C(2\sigma) \ne Y$ and $F(x^0) \cap (y^0 - C(2\sigma)) = \{y^0\}$. Then for each $x \in U \cap X_0$ and each $y \in F(x) \cap (y^0 - C(\sigma))$ it holds $\|y - y^0\| \le (L/\sigma)\,\|x - x^0\|$.*

Proof Denote by y' a orthogonal projection of y on $\mathrm{cl}\,(y^0 - C(2\sigma))^c$, where $(y^0 - C(2\sigma))^c := Y \setminus (y^0 - C(2\sigma))$. Let us underline, that the set $\mathrm{cl}\,(y^0 - C(2\sigma))^c$ is not empty, due to the property $C(2\sigma) \ne Y$. Then

$$\|y - y'\| \le D(y - y^0, F(x^0) + C) \le L\,\|x - x^0\|\,.$$

On the other hand it holds $\|y - y'\| \ge \sigma\|y - y^0\|$. To show this inequality it is enough to observe that $y - y^0 \in -C(\sigma)$ implies

$$\mathrm{cone}\,(y - y^0)(\sigma) \subset -C(\sigma)(\sigma) \subset -C(2\sigma)$$

and $-C(2\sigma)$ does not contain $y' - y^0$ in its interior. For the inclusion $C(\sigma)(\sigma) \subset C(2\sigma)$ we have applied Lemma 2.1. The above inequalities give in consequence $\|y - y^0\| \le (L/\sigma)\,\|x - x^0\|$. □

Theorem 2.1 *Let in svp (1) with $Y = \mathbb{R}^m$ and closed (not necessarily convex) cone C the svf $F : X_0 \rightsquigarrow Y$ be locally C-Lipschitz. Suppose that (x^0, y^0), $y^0 \in F(x^0)$, is a i-minimizer for (1). Then (x^0, y^0) is also a p-minimizer of (1) (for this conclusion X_0 need not be locally convex at x^0).*

Proof The assumption that (x^0, y^0) is i-minimizer implies that there exists a a neighbourhood U of x^0 and constants $A > 0$ and $\sigma > 0$, such that $D(F(x) - y^0, -C) \geq A \,\|x - x^0\|$ for $x \in U$, and $F(x^0) \cap (y^0 - C(2\sigma)) = \{y^0\}$. Diminishing eventually σ, we may accept that $C(2\sigma) \neq Y$. We suppose also that in U the Lipschitz condition (5) is satisfied.

Assume now that (x^0, y^0) is not a p-minimizer of (1). Therefore there exist sequences $x^k \to x^0$, $y^k \in F(x^k)$ and $\varepsilon_k \to 0^+$ such that $x^k \in U$, $\varepsilon_k < \sigma$, $y^k \in y^0 - \operatorname{int} C(\varepsilon_k)$. The latter inclusion gives in particular $y^k \neq y^0$. Now with regard to $y^k \in y^0 - \operatorname{int} C(\varepsilon_k) \subset y^k \in y^0 - C(\sigma)$ and applying Lemma 2.2 we get

$$D(F(x^k) - y^0, -C) \leq D(y^k - y^0, -C) \leq \varepsilon_k \,\|y^k - y^0\| \leq \varepsilon_k \,\frac{L}{\sigma}\,\|x^k - x^0\|\,.$$

From this chain of inequalities we get $x^k \neq x^0$, since otherwise we would have the contradictory inequalities $0 < \|y^k - y^0\| \leq 0$. However, if $x^k \neq x^0$ from the inequalities

$$A\,\|x^k - x^0\| \leq D(F(x^k) - y^0, -C) \leq \varepsilon_k \,\frac{L}{\sigma}\,\|x^k - x^0\|$$

we get $0 < A \leq \varepsilon_k \,(L/\sigma)$. A passing to a limit with $k \to \infty$ gives $0 < A \leq 0$, a contradiction. $\square$

3 First-order optimality conditions

In this section we establish first-order optimality conditions for svp (1), that is conditions expressed in first-order derivatives of svf F. More precisely, we use Dini-directional derivatives. We start with the needed definitions.

Recall [1] that for svf $\Phi : T_0 \to Y$ defined on a subset T_0 of the topological space T the upper limit $\operatorname{Limsup}_{t \to t_0} \Phi(t)$ and the lower limit $\operatorname{Liminf}_{t \to t_0} \Phi(t)$ are defined respectively by

$$\operatorname{Limsup}_{t \to t_0} \Phi(t) = \{y \in Y \mid \liminf_{t \to 0^+} d(y, \Phi(t)) = 0\},$$

$$\operatorname{Liminf}_{t \to t_0} \Phi(t) = \{y \in Y \mid \lim_{t \to 0^+} d(y, \Phi(t)) = 0\}\,.$$

Let us underline that writing here $t \to t_0$ we accept that t varies in T_0 and $t_0 \in \operatorname{cl} T_0$ but not necessary $t_0 \in T_0$. We say that the limit $\operatorname{Lim}_{t \to 0^+} \Phi(t)$ exists if the upper and the lower limit coincide. Then their common value is denoted by $\operatorname{Lim}_{t \to 0^+} \Phi(t) = \operatorname{Limsup}_{t \to t_0} \Phi(t) = \operatorname{Liminf}_{t \to t_0} \Phi(t)$. The svf Φ is said to be respectively lower semi-continuous (lsc) at t_0 if $t_0 \in T_0$ and $\Phi(t_0) \subset \operatorname{Liminf}_{t \to 0^+} \Phi(t)$, upper semi-continuous (usc) at t_0 if $t_0 \in T_0$ and $\Phi(t_0) \supset \operatorname{Limsup}_{t \to 0^+} \Phi(t)$, continuous at t_0 if $t_0 \in T_0$ and $\Phi(t_0) = \operatorname{Lim}_{t \to 0^+} \Phi(t)$.

We define the Dini-derivative of the svf $F : X_0 \to Y$ at (x^0, y^0), $y^0 \in F(x^0)$, in the feasible direction $u \in X$, as the upper limit

$$F'(x^0, y^0; u) = \operatorname*{Limsup}_{t \,\to\, +0} \frac{1}{t}\left(F(x^0 + tu) - y^0\right)\,.$$

Theorem 3.1 (Necessary Conditions, w-minimizers) *Consider svp (1) with $F : X_0 \rightsquigarrow Y$ and C closed convex cone. Let (x^0, y^0), $y^0 \in F(x^0)$, be a w-minimizer. Then*

$$\forall\, u \in X_0(x^0) : F'(x^0, y^0; u) \cap (-\mathrm{int}\, C) = \emptyset\,. \tag{6}$$

Proof Fix $u \in X_0(x^0)$. From the definition of a w-minimizer we have $(1/t)(F(x^0+tu)-y^0) \notin -\mathrm{int}\, C$ for all sufficiently small $t > 0$. Passing to a limit with $\mathrm{Limsup}_{t\to 0^+}$ we get (6). □

Remark 3.1 Theorem 3.1 can be reformulated exchanging condition (6) with

$$\forall\, u \in X_0(x^0) : \forall\, \bar{y}^0 \in F'(x^0, y^0; u) : \exists\, \bar{\xi}^0 \in C' \setminus \{0\} : \langle \bar{\xi}^0,\, \bar{y}^0\rangle \geq 0\,. \tag{7}$$

Conditions like (6) expressed directly in terms of the directional derivative are called primal, and their equivalent formulations in terms of the positive polar cone are called dual. Therefore, Theorem 3.1 gives necessary conditions in primal form a pair (x^0, y^0) to be a w-minimizer of svp (1). The equivalent formulation in which condition (6) is replaced by (7) gives necessary conditions in dual form the pair (x^0, y^0) to be a w-minimizer of svp (1). Let us underline that the theorem in primal form is valid also when C is not convex, while the dual form fails when C is not convex. Finally, turn attention that the theorem is proved for X_0 not necessary locally convex at x^0 and Y not necessary finite dimensional.

Next in Theorem 3.2 we characterize the i-minimizers for problems with locally C-Lipschitz functions. In advance, we establish some properties of the locally C-Lipschitz function.

Proposition 3.1 *Let the svf $F : X_0 \rightsquigarrow Y$ be C-Lipschitz with constant L on $X_0 \cap (x^0 + r\, \mathrm{cl}\, B_X)$, $r > 0$, with respect to the closed cone $C \subset Y$, and let $y^0 \in F(x^0)$.*

Then for $0 < t < r/\max(\|u\|,\, \|v\|)$ and $x^0 + tv \in X_0$, $x^0 + tu \in X_0$ it holds

$$\frac{1}{t}\left(F(x^0+tv) - y^0\right) \subset \frac{1}{t}\left(F(x^0+tu) - y^0\right) + C + L\,\|v-u\|\, \mathrm{cl}\, B_Y\,. \tag{8}$$

In particular for $u = 0$, $0 < t < r/\|v\|$ and $x^0 + tv \in X_0$ we get

$$\frac{1}{t}\left(F(x^0+tv) - y^0\right) \subset \frac{1}{t}\left(F(x^0) - y^0\right) + C + L\,\|v\|\, \mathrm{cl}\, B_Y\,. \tag{9}$$

For the derivative of F at $u = 0$ we have $F'(x^0, y^0; 0) \subset \mathrm{cl}\,\mathrm{cone}\,(F(x^0) - y^0)$. In consequence for $v \in X_0(x^0)$ we have

$$F'(x^0, y^0; v) \subset \mathrm{cl}\,\mathrm{cone}\,(F(x^0) - y^0) + C + L\,\|v\|\, \mathrm{cl}\, B_Y\,. \tag{10}$$

Proof The Lipschitz property of F gives $F(x^0+tv) \subset F(x^0+tu) + C + L\,\|u-v\|\,\mathrm{cl}\, B_Y$ from which after obvious recasting it follows (8). Inclusion (8) implies immediately (9).

Obviously for $t > 0$ we have $(1/t)(F(x^0) - y^0) \subset \mathrm{cone}\,(F(x^0) - y^0)$. This inclusion gives $F'(x^0, y^0; 0) \subset \mathrm{cl}\,\mathrm{cone}\,(F(x^0) - y^0)$. Now (9) gives for t sufficiently small

$$\frac{1}{t}\left(F(x^0+tv) - y^0\right) \subset \mathrm{cl}\,\mathrm{cone}\,(F(x^0) - y^0) + C + L\,\|v\|\, \mathrm{cl}\, B_Y\,.$$

whence we get straightforward (10). □

Inclusion (8) opens the question, whether for the derivative of a locally C-Lipschitz $F : X_0 \rightsquigarrow Y$ with Y finite dimensional and $u,\, v \in X_0(x^0)$ the following inclusion is true

$$F'(x^0, y^0; v) \subset F'(x^0, y^0; u) + C + L\, \|v - u\| \operatorname{cl} B_Y\,. \tag{11}$$

In opposite to our expectations, Example 3.1 below demonstrates that (11) is not true. It shows also, that the derivative $F'(x^0, y^0; u)$ could be empty. Let us turn attention, that in Proposition 3.2 below it is proved that this is not the case, when Lipschitz instead of C-Lipschitz svf is considered. By the way, by definition svf F is Lipschitz if and only if it is C-Lipschitz with respect to the trivial cone $C = \{0\}$. Therefore, the conclusion of Proposition 3.1 remains true for Lipschitz svf with replacement of C by $\{0\}$.

Example 3.1 *Consider the function*

$$f : \mathbb{R} \to \mathbb{R}^2\,, \quad f(x) = \begin{cases} (|\,x|\,,\, 1/|\,x|) & , \quad x \neq 0\,, \\ (0,\, 0) & , \quad x = 0\,. \end{cases}$$

Let $C = \{(y_1,\, y_2) \in \mathbb{R}^2 \mid y_1 \geq 0\}$. Then f is C-Lipschitz with constant $L = 1$, but its derivative does not satisfy inclusion (11) for $x^0 = 0$, $y^0 = (0,\, 0)$, $v = 0$, $u \neq 0$. In fact, we have $f'(x^0,\, 0) = \{(0,\, 0)\}$ and $f'(x^0,\, u) = \emptyset$ for $u \neq 0$. (for vvf f we prefer to write $f'(x^0, u)$ instead of $f'(x^0, y^0; u)$ for the derivative at (x^0, y^0), $y^0 = f(x^0)$, in direction u).

The C-Lipschitz property of f in this example is a consequence of the inequality $|\,x_2| \leq |\,x_1| + |\,x_2 - x_1|$. For $v = 0$ we have $(1/t)(f(x^0 + tv) - y^0) = (0,\, 0)$, whence $f'(x^0, v) = \{(0,\, 0)\}$. For $u \neq 0$ we have $(1/t)(f(x^0 + tu) - y^0) = (|u|,\, 1/(t^2|u|))$, whence $f'(x^0, u) = \emptyset$. Inclusion (10) is not satisfied, otherwise we would have $\{(0,\, 0)\} \subset \emptyset + C + \|u\|\, B_y = \emptyset$, a contradiction.

Proposition 3.2 *Let the svf $F : X_0 \rightsquigarrow Y$ with $Y = \mathbb{R}^m$ be Lipschitz with constant L on $X_0 \cap (x^0 + r \operatorname{cl} B_X)$, $r > 0$, and let $y^0 \in F(x^0)$. Then for all $u,\, v \in X_0(x^0)$ we have $F'(x^0, u^0; u) \cap (L\,\|u\| \operatorname{cl} B_Y) \neq \emptyset$ and the following inclusion is satisfied.*

$$F'(x^0, y^0; v) \subset F'(x^0, y^0; u) + L\, \|v - u\| \operatorname{cl} B_Y\,. \tag{12}$$

Proof First we prove (12). Let $\bar{y}_v^0 \in F'(x^0, y^0; v)$. Therefore $y_v^0 = \lim_k \bar{y}_v^k$, where $\bar{y}_v^k = (1/t_k)(y_v^k - y^0)$, $y_v^k \in F(x^0 + t_k v)$. The Lipschitz property of F gives that there exist sequences $y_u^k \in F(x^0 + t_k u)$ and $b^k \in \operatorname{cl} B_Y$ such that $y_v^k = y_u^k + L t_k\, \|v - u\|\, b^k$, whence putting $\bar{y}_u^k = (t_k)(y_u^k - y^0)$ we get $\bar{y}_v^k = \bar{y}_u^k + L\|v - u\|\, b^k$. The boundedness of the sequences $\bar{y}_v^k$ and b^k imply the boundedness of $\bar{y}_u^k$. Now, passing to a subsequence, we may assume that $\bar{y}_u^k \to \bar{y}_u^0 \in F'(x^0, y^0; u)$ and $b^k \to b^0 \in \operatorname{cl} B_Y$. A passing to a limit in the above inequality gives that $\bar{y}_v^0 = \bar{y}_u^0 + L\, \|v - u\|\, b^0$, which proves (12).

Now we prove that that $F'(x^0, u^0; u) \cap (L\, \|u\| \operatorname{cl} B_Y) \neq \emptyset$. Turn attention, that together with (12) we have proved that if $y_v \in F'(x^0, u^0; v)$ for some v, then there exists $y_u \in F'(x^0, u^0; u)$ such that $\|y_u - y_v\| \leq L\, \|u - v\|$. Since obviously $0 \in F'(x^0, u^0; 0)$, therefore there exists $y_u \in F'(x^0, u^0; u)$ such that $\|y_u\| \leq L\, \|u\|$. $\square$

The next theorem characterizes the i-minimizers of svp with locally C-Lipschitz svf.

Theorem 3.2 (Sufficient Conditions, i-minimizers) *Consider svp (1) with $Y = \mathbb{R}^m$, $C \neq Y$ a closed convex cone and svf $F : X_0 \rightsquigarrow Y$ being locally C-Lipschitz. Suppose that (x^0, y^0), $y^0 \in F(x^0)$, is such that X_0 is locally convex at x^0 with closed cone $X_0(x^0)$ of feasible directions, $y^0 \in p\text{-}Min_C F(x^0)$ and*

$$\forall\, u \in X_0(x^0) \setminus \{0\} : F'(x^0, y^0; u) \cap (-C) = \emptyset\,. \tag{13}$$

Then (x^0, y^0) is a i-minimizer for (1).

Proof Assume in the contrary, that x^0 is not a i-minimizer. Choose a monotone decreasing sequence $\varepsilon_k \to 0^+$ The made assumption gives that there exist sequences $t_k \to 0^+$, $u^k \in X_0(x^0) \cap S_X$, such that $D(F(x^0 + t_k u^k) - y^0, -C) < \varepsilon_k t_k$. We may assume that F is C-Lipschitz with constant $L > 0$ on $x^0 + r \operatorname{cl} B_Y$, $X_0 \cap (x^0 + r \operatorname{cl} B_Y)$ is convex, and $t_k < r$. Passing to a subsequence, we may assume also that $u^k \to u^0$. The closedness of $X_0(x^0)$ gives $u^0 \in X_0(x^0) \cap S_X$.

The Lipschitz property gives $D(F(x^0 + t_k u^0) - y^0, -C) < \varepsilon_k t_k + L \|u^k - u^0\| \, t_k$, which due to the positive homogeneity of $D(\cdot, -C)$ can be written as

$$D\Big(\frac{1}{t_k}\left(F(x^0 + t_k u^0) - y^0\right), -C\Big) < \varepsilon_k + L \|u^k - u^0\| \,.$$

Let $y^k \in F(x^0 + t_k u^0)$ be such that $D(\bar{y}^k, -C) < \varepsilon_k + L\|u^k - u^0\|$, where $\bar{y}^k = (1/t_k)(y^k - y^0)$. The sequence $\bar{y}^k$ is bounded, which follows from the following reasoning. Since $y^0 \in p\text{-}Min_C F(x^0)$, there exists $\sigma > 0$, such that $F(x^0) \cap (y^0 - C(2\sigma)) = \{y^0\}$. Due to $C \neq Y$, eventually diminishing σ, we may assume that $C(2\sigma) \neq Y$. Let k be such that $\varepsilon_k + L \|u^k - u^0\| < L$. Then $\|\bar{y}^k\| \leq L/\sigma$. Indeed, assume in the contrary that $\|\bar{y}^k\| > L/\sigma$, or equivalently $\|y^k - y^0\| > (L/\sigma)\, t_k$. We have

$$D(y^k - y^0, -C) \leq L t_k \frac{1}{\|y^k - y^0\|} \|y^k - y^0\| \leq \sigma \|y^k - y^0\| \,.$$

This inequality shows that $y^k - y^0 \in -C(\sigma)$, whence according to Lemma 2.2 we get

$$\|y^k - y^0\| \leq \frac{L}{\sigma} \|(x^0 + t_k u^0) - x^0\| = \frac{L}{\sigma}\, t_k \,,$$

a contradiction.

So, we proved that the sequence $\bar{y}^k$ is bounded, and more, it holds $\|\bar{y}^k\| \leq L/\sigma$ for all sufficiently large k. Passing to a subsequence, we may assume that $\bar{y}^k \to \bar{y}^0$, whence $\|\bar{y}^0\| \leq L/\sigma$ and $\bar{y}^0 \in F'(x^0, y^0; u^0)$. In other words $\bar{y}^0 \in F'(x^0, y^0; u^0) \cap (L/\sigma) \operatorname{cl} B_Y$. This set is compact (recall that $F'(x^0, y^0; u)$ is closed as a consequence of the general properties of the upper limit, see the representation in [1, page 41]). From the compactness and the property $F'(x^0, y^0; u) \cap (-C) = \emptyset$ we have

$$D(\bar{y}^0, -C) \geq D(F'(x^0, y^0; u) \cap \frac{L}{\sigma} \operatorname{cl} B_Y, -C) > 0 \,.$$

On the other hand, taking a limit in the inequality $D(\bar{y}^k, -C) \leq \varepsilon_k + L \|u^k - u^0\|$, we get $D(\bar{y}^0, -C) \leq 0$, a contradiction. $\square$

Remark 3.2 Theorem 3.2 gives sufficient conditions in primal form a point (x^0, y^0), to be a i-minimizer for svp (1). The equivalent dual formulation of condition (13) is

$$\forall\, u \in X_0(x^0) \setminus \{0\} : \forall\, \bar{y}^0 \in F'(x^0, y^0; u) : \exists\, \bar{\xi}^0 \in C' \setminus \{0\} : \langle \bar{\xi}^0,\, \bar{y}^0 \rangle > 0 \,. \tag{14}$$

Let us underline that the theorem in primal form is valid also when C is not convex, while the dual form fails for non-convex C.

The importance of the notion of a i-minimizer we see in the possibility to revert Theorem 3.2, i. e. this is the appropriate notion of optimality being characterized by the sufficient conditions. Such a reversal we propose in the next theorem. In fact we prove the reversal under more general assumptions than those of Theorem 3.2.

Theorem 3.3 (Necessary Conditions, i-minimizers) *Consider svp (1) with C closed convex cone and svf $F : X_0 \rightsquigarrow Y$. Suppose that (x^0, y^0), $y^0 \in F(x^0)$, is a i-minimizer for (1). Then $y^0 \in p\text{-}Min_C F(x^0)$ and condition (13) holds.*

Proof Assume that (x^0, y^0) is a i-minimizer. Then $y^0 \in p\text{-}Min_C F(x^0)$ according to the definition of an isolated minimizer. Because of the positive homogeneity of $F'(x^0, y^0; \cdot)$ it suffices to prove (13) for $u \in (X_0(x^0) \setminus \{0\}) \cap S_X$. In this case the definition of a i-minimizer gives $D((1/t)(F(x^0 + tu) - y^0), -C) \geq A > 0$, whence passing to a limit with $\mathrm{Limsup}_{t \to 0^+}$ we get $D((F'(x^0, y^0; u) - y^0), -C) \geq A > 0$. In particular $F'(x^0, y^0; u) \cap (-C) = \emptyset$. □

Example 3.2 *Consider the vvp (3), where the function f and the cone C are defined as in Example 3.1. Then the function f is C-Lipschitz, and $x^0 = 0$ is a i-minimizer, which can be established on the base of Theorem 3.2 (in the case of a vvp with a vvf f we prefer to say that x^0 is a i-minimizer instead of $(x^0,\, y^0)$, $y^0 = f(x^0)$, is a i-minimizer).*

Turn attention, that in the case when F is single-valued, the condition $(x^0, y^0) \in p\text{-}Min_C F(x^0)$ is trivially satisfied. In Example 3.1 it was shown that the considered there function is C-Lipschitz. Condition (13) is satisfied, since for $u \neq 0$ it holds $f'(x^0, u) = \emptyset$.

After the above example we can make some comments. As it was said in the introduction, the purpose of the present paper is to generalize to set-valued functions the optimality conditions from [9] confining only to unconstrained problem (the consideration of constrained problems could be a subject of another paper). In [9] locally Lipschitz data are considered. When extending the results to set-valued functions, one realizes that the i-minimizers of the svp with functions $x \rightsquigarrow F(x)$ and $x \rightsquigarrow F(x) + C$ coincide. It is more convenient to work with the function $F + C$ for the following reason. Two values $y^1 \in F(x^1)$ and $y^2 \in F(x^2)$ not belonging to the efficient frontiers may differ much, but when optimal solutions are concerned, of importance are only the values from the efficient frontiers. Passing from svf F to svf $F + C$ we gain the advantage, that the vicinity of the efficient frontiers of $F(x^1)$ and $F(x^2)$ gives vicinity of the image sets $F(x^1) + C$ and $F(x^2) + C$. With this imagination, it is not difficult to discover, that in the set-valued optimization the C-Lipschitz property should be of more importance, than the Lipschitz property. The characterization of i-minimizers from Theorem 3.2 is patterned on the similar result for locally Lipschitz single-valued functions from [9]. Now we suddenly reveal, that the result from [9] is true not only for Lipschitz, but also for C-Lipschitz functions. In particular we gain a tool to treat problems like the one from Example 3.1, which was not possible within the framework of the result from [9]. Probably, confining to the single-valued case, it would not be so easy to come to the concept of a C-Lipschitz function and to discover the generalization of the result of [9] from locally Lipschitz to locally C-Lipschitz functions. Still, it remains the problem to investigate closer the relation between locally C-Lipschitz and locally Lipschitz functions. The next proposition relates to this problem in the case of a single-valued function.

Proposition 3.3 *Let the vvf $f : X_0 \to Y$ with $Y = \mathbb{R}^m$ be C-Lipschitz with constant L with respect to the pointed closed convex cone $C \subset Y$. Then f is also Lipschitz.*

Proof Take x^0, $x^1 \in X_0$ and put $y^0 = f(x^0)$, $y^1 = f(x^1)$, $r = L\,\|x^1 - x^0\|$. The C-Lipschitz property gives $y^1 \in y^0 + C + r\,\mathrm{cl}\,B_Y$ and $y^0 \in y^1 + C + r\,\mathrm{cl}\,B_Y$. The second inclusion is transformed into $y^1 \in y^0 - C + r\,\mathrm{cl}\,B_Y$. Therefore $y^1 - y^0 \in (C + \mathrm{cl}\,B_Y) \cap (-C + r\,\mathrm{cl}\,B_Y)$.

Recall that C pointed means $C \cap (-C) = \{0\}$. If C is pointed closed convex cone, as an application of the Separation Theorem and using compactness arguments, it can be shown that there exists $\sigma > 0$, such that $C(2\sigma)$ is still a pointed closed convex cone. Now we show that $\|y^1 - y^0\| \le (L/\sigma)\, \|x^1 - x^0\|$, which is the desired Lipschitz property. Assume in the contrary, that $\|y^1 - y^0\| > (L/\sigma)\, \|x^1 - x^0\|$. A consequence of $y^1 - y^0 \in -C + r \,\mathrm{cl}\, B_Y$ is $D(y^1 - y^0, -C) \le r$, which together with the above inequality gives $D(y^1 - y^0, -C) \le \sigma \|y^1 - y^0\|$, which together with the assumed inequality gives $D(y^1 - y^0, -C) \le \sigma \|y^1 - y^0\|$, whence $y^1 \in y^0 - C(\sigma)$. Now, taking into account $y^1 - y^0 \in C + r \,\mathrm{cl}\, B_Y$ and applying Lemma 2.2 we get $\|y^1 - y^0\| \le (L/\sigma)\, \|x^1 - x^0\|$, a contradiction with the made assumption. □

For set-valued functions the following result is true. We skip the proof, since it maintains the same idea as the proof of Proposition 3.3.

Proposition 3.4 *Let the svf $F : X_0 \rightsquigarrow Y$ with $Y = \mathbb{R}^m$ be C-Lipschitz with constant L with respect to the closed (not necessary convex) cone $C \subset Y$. Suppose that there exists a closed (not necessary convex) cone $T \ne \{0\}$, a selection $f : X_0 \to Y$, $f(x) \in F(x)$, and a constant $\sigma > 0$, such that $F(x) \subset f(x) + T$ and $(T \setminus \{0\}) \cap (-C(2\sigma)) = \emptyset$. Then F is Lipschitz with constant L/σ.*

4 Isolated minimizers and proper efficiency

In this section as an application of Theorem 3.2 we discuss the reversal of Theorem 2.1. In general even under Lipschitz type conditions a p-minimizer of (1) need not be a i-minimizer, which is seen in the next example.

Example 4.1 *Consider vvp (3) with $f : \mathbb{R} \to \mathbb{R}^2$, $f(x) = (x^2,\, x^2)$ and $C = \mathbb{R}^2_+$. Then f is locally Lipschitz, and $x^0 = 0$ is a p-minimizer, but not a i-minimizer.*

In this example $f'(x^0, u) = \{(0,\, 0)\} \in -C$, which is the crucial moment, as the next theorem shows.

Theorem 4.1 *Let in svp (1) with $Y = \mathbb{R}^m$ and closed (not necessarily convex) cone C the svf $F : X_0 \rightsquigarrow Y$ be locally C-Lipschitz. Suppose that (x^0, y^0), $y^0 \in F(x^0)$, is such that X_0 is locally convex at x^0 with closed cone $X_0(x^0)$ of feasible directions, and is a p-minimizer, which has the property*

$$\forall\, u \in X_0(x^0) \setminus \{0\} : 0 \notin F'(x^0, y^0; u)\,.$$

Then (x^0, y^0) is a i-minimizer for (1).

Proof Fix $u \in X_0(x^0) \setminus \{0\}$. Since (x^0, y^0) is a p-minimizer, there exists $a > 0$, such that $F(x^0 + tu) \cap (y^0 - \mathrm{int}\, C(a)) = \emptyset$ for all sufficiently small $t > 0$. This gives $(1/t)(F(x^0 + tu) - y^0) \cap (-\mathrm{int}\, C(a)) = \emptyset$. Passing to a limit with $\mathrm{Limsup}_{t \to 0^+}$ we get $F'(x^0, y^0; u) \cap (-\mathrm{int}\, C(a)) = \emptyset$, whence taking into account that $-C \setminus \{0\} \subset \mathrm{int}\, C(a)$ and $0 \notin F'(x^0, y^0; u)$ we get $F'(x^0, y^0; u) \cap (-C) = \emptyset$. As we know, (x^0, y^0) p-minimizer implies $y^0 \in p\text{-}Min_C F(x^0)$. Therefore the assumptions of Theorem 3.2 are satisfied, whence we get the conclusion that (x^0, y^0) is a i-minimizer. □

5 Optimality under convexity type conditions

In the textbooks of classical analysis usually first necessary and thereafter sufficient optimality conditions are established. With Theorems 3.1 and 3.2 we follow the same order. The imagination that the sufficient conditions concern a more qualified concept of optimality, namely the concept of i-minimizer, revealed the possibility to revert Theorem 3.2, obtaining in such a way not only sufficient, but also necessary conditions a point (x^0, y^0) to be a i-minimizer. Now the natural question arises, whether similarly a characterization of the w-minimizers can be obtained reverting Theorem 3.1. In general such a reversal does not hold. However like in convex analysis this is possible under convexity type conditions. We establish such a result in Theorem 5.1 below. In convex analysis usually convexity type conditions are associated to global minimizers. By analogy, we propose a result, which concerns global w-minimizers. First we give some definitions.

The pair (x^0, y^0), $y^0 \in F(x^0)$, is said to be a global w-minimizer for svp (1) if for every $x \in X_0$ it holds $F(x) \cap (y^0 - \operatorname{int} C) = \emptyset$. Similarly, one can define global versions of all the optimality concepts introduced in Section 2.

We say that the svf $F : X_0 \rightsquigarrow Y$ is C-convex-along-rays at (x^0, y^0) if the set X_0 is star shaped at x^0 and $(1-t)y^0 + tF(x) \subset F((1-t)x^0 + tx) + C$ for all $x \in X_0$ and $0 < t < 1$. Recall that X_0 is star shaped at x^0 if $(1-t)x^0 + tx \subset X_0$ for all $x \in U \cap X_0$ and $0 < t < 1$. The concept of a convex-along-rays scalar-valued function is introduced in Rubinov [26] and is used there for the purposes of abstract convexity and global optimization.

We need also the following lemma.

Lemma 5.1 *Let $C \subset \mathbb{R}^m$ be pointed closed convex cone. Then for any a^1, $a^2 \in \mathbb{R}^m$ the set $(a^1 - C) \cap (a^2 + C)$ is bounded.*

Proof Subtracting a^1 from the two sets in the intersection and putting $a = a^2 - a^1$, we see that the conclusion is equivalent to the statement: $(-C)\cap(a+C)$ bounded for arbitrary $a \in Y$. If the intersection is empty, then the latter is true. Let now $-C \cap (a + C) \neq \emptyset$. Then $-c^1 = a + c^2$ for some c^1, $c^2 \in C$, whence $a = -c^1 - c^2 \in -C$. The condition $C \subset \mathbb{R}^m$ pointed closed convex cone implies that there exists $\sigma > 0$ such that the cone $C(2\sigma)$ is closed and convex, whence $-C(2\sigma)\cap C = \{0\}$. Let $y \in -C\cap(a+C)$. Then $y - a \in C$ and

$$D(y, C) = D(-y, -C)$$
$$= \sup_{\xi\in C',\, \|\xi\|=1} \langle \xi,\, -(y-a) - a\rangle \le \sup_{\xi\in C',\, \|\xi\|=1} \langle \xi,\, -a\rangle \le \|a\| \,.$$

Therefore $(a + C) \cap (-C) \subset C + \|a\|\, B_Y$. We face the same situation like in Lemma 2.2, with $F : \mathbb{R}^m \rightsquigarrow \mathbb{R}^m$, $F(x) = x + C$, $(x^0,\, y^0) = (0,\, 0)$. The svf F is C-Lipschitz with constant $L = 1$. Now Lemma 2.2 gives that for $x = a$, $y \in (a + C) \cap (-C) \subset (a + C) \cap (-C(\sigma))$ it holds $\|y\| \le \|a\|/\sigma$. Consequently $(a + C) \cap (-C) \subset (\|a\|/\sigma)\, B_Y$. □

Theorem 5.1 (Sufficient Conditions, w-minimizers) *Consider svp (1) with $Y = \mathbb{R}^m$ and $C \subset Y$ pointed closed convex cone. Suppose that it holds (x^0, y^0), $y^0 \in w\text{-}\mathrm{Min}_C F(x^0)$, is such that X_0 is star shaped at x^0, $F : X_0 \rightsquigarrow Y$ is C-convex-along-rays at (x^0, y^0), and condition (6) is satisfied. Suppose also that for each direction $u \in X_0(x^0) \setminus \{0\}$ there exists a vector $g_u \in Y$ such that $F(x^0 + tu) \subset y^0 + tg_u + C$. Then (x^0, y^0) is a global w-minimizer for (1).*

Proof Assume in the contrary, that (x^0, y^0) is not a global w-minimizer. Then there exists a pair (x^1, y^1) such that $x^1 \in X_0$ and $y^1 \in F(x^1) \cap (y^0 - \operatorname{int} C)$. From the assumption $y^0 \in w\text{-}\mathrm{Min}_C F(x^0)$ it follows that $x^1 \neq x^0$. Put $u = x^1 - x^0$. Since X_0 is star shaped at x^0 and $x^1 \neq x^0$, we have $u \in X_0(x^0) \setminus \{0\}$. We define

$$\Phi(t) = \frac{1}{t}\left(F(x^0 + tu) - y^0\right) \cap (y^1 - y^0 - C)\,.$$

We prove that $\Phi(t) \neq \emptyset$ for $0 < t < 1$. Indeed, from F C-convex-along-rays we have

$$(1-t)y^0 + ty^1 \in (1-t)y^0 + tF(x^1) \subset F((1-t)x^0 + tx^1) + C = F(x^0 + tu) + C\,,$$

whence $(1-t)y^0 + ty^1 = y + c$ for some $y \in F(x^0 + tu)$ and $c \in C$. Consequently

$$\frac{1}{t}\left(y - y^0\right) \in y^1 - y^0 - \frac{1}{t}\, c \in y^1 - y^0 - C\,,$$

and finally $\Phi(t) \neq \emptyset$. According to the made assumptions we have also

$$\Phi(t) \subset \frac{1}{t}\left(F(x^0 + tu) - y^0\right) \subset g_u + C\,.$$

Therefore $\Phi(t) \subset (y^1 - y^0 - C) \cap (g_u + C)$. The intersection on the right hand side is bounded according to Lemma 5.1, whence $\Phi_0 := \mathrm{Limsup}_{t\to 0^+}\Phi(t) \neq \emptyset$. Thus, $\Phi_0 \subset y^1 - y^0 - C \subset -\mathrm{int}\, C$. At the same time $\Phi(t) \subset (1/t)(F(x^0 + tu) - y^0)$, whence passing to a limit with $\mathrm{Lisup}_{t\to 0^+}$ we get $\Phi_0 \subset F'(x^0, y^0; u)$. Therefore $F'(x^0, y^0; u) \cap (-\mathrm{int}\, C) \neq \emptyset$, a contradiction to (6). $\square$

The next example shows that the condition $F(x^0 + tu) \subset y^0 + tg_u + C$ is important for the validity of Theorem 5.1.

Example 5.1 *Consider vvp (3) with $X = X_0 = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, and*

$$f : \mathbb{R} \to \mathbb{R}^2, \quad f(x) = (x,\, -\sqrt{|x|})\,.$$

Then f is C-convex-along-rays at $(x^0,\, y^0)$, where $x^0 = 0$ and $y^0 = (0,\, 0)$. Condition (6) is satisfied. At the same time $(x^0,\, y^0)$ is not a w-minimizer.

To prove the convexity property of f we must check the inclusion $(1-t)y^0 + tf(x) \in f((1-t)x^0 + tx) + \mathbb{R}^2_+$, for each $x \in \mathbb{R}$ and $0 < t < 1$. This follows from $tf(x) - f(tx) = (0,\, (\sqrt{t} - t)\,\sqrt{|x|}) \in \mathbb{R}^2_+$.

For the derivative of f we have

$$f'(x^0, u) = \begin{cases} \{(0,\,0)\} & , \quad u = 0\,, \\ \emptyset & , \quad u \neq 0\,, \end{cases}$$

The second row follows from $(1/t)(f(x^0 + tu) - y^0) = (u, -\sqrt{|u|/t})$. From here obviously for $u \neq 0$ it holds $f'(x^0, u) \cap (-\mathrm{int}\,\mathbb{R}^2_+) = \emptyset$.

For each $x < 0$ we have $f(x) = (x,\, -\sqrt{|x|}) \in -\mathrm{int}\,\mathbb{R}^2_+$. Therefore x^0 is not (even local) w-minimizer.

The next example illustrates an application of Theorem 5.1.

Example 5.2 *Let $X = X_0 = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$ and $F : X_0 \rightsquigarrow \mathbb{R}^2$ be given by*

$$F(x) = \begin{cases} [0,\,1] \times [0,\,1] & , \quad x \neq 0, \\ ([-1,\,0] \times \{0\}) \cup (\{0\} \times [-1,\,0]) & , \quad x = 0. \end{cases}$$

Put $x^0 = 0$ and $y^0 = (0,\,0)$. Then $(x^0,\, y^0)$ is a global w-minimizer, which can be established on the base of Theorem 5.1.

To show the C-convexity-along-rays of F at (x^0, y^0) we must check that $tF(x) \subset F(tx) + \mathbb{R}^2_+$ for $0 < t < 1$. For $x \neq x^0$ this is the true inclusion $[0,\,1] \times [0,\,1] \subset ([0,\,1] \times [0,\,1]) + \mathbb{R}^2_+$. For $x = x^0$ the validity follows from the true inclusion $[-t,\,0] \subset [0,\,0]$.

Easy calculations give that

$$F'(x^0, y^0; u) = \begin{cases} \mathbb{R}^2_+ & , \quad u \neq 0, \\ (\mathbb{R}_- \times \{0\}) \cup (\{0\} \times \mathbb{R}_-) & , \quad u = 0, \end{cases}$$

whence it is obvious that condition (6) is satisfied. Further for $u \neq 0$ the vector $g_u = (0,\,0)$ satisfies the condition from Theorem 5.1. In consequence we get that (x^0, y^0) is a global w-minimizer.

6 Final remarks

In our opinion the concept of an isolated minimizer for set-valued optimization is a new one. As for w-minimizers, the following result holds.

Theorem 6.1 (Jahn, Rauh [19]) *Let in the svp (1) C be a convex cone with nonempty interior, let X_0 be a convex set and let $F : X_0 \rightsquigarrow Y$ be C-convex. Let the contingent epiderivative $DF(x^0, y^0)$ exists at $x^0 \in X_0$, $y^0 \in F(x^0)$. Then the pair (x^0, y^0) is a w-minimizer for (1) if and only if $DF(x^0, y^0)(x - x^0) \notin -\operatorname{int} C$ for all $x \in X_0$.*

The contingent epiderivative $DF(x^0, y^0)$ of the svf $F : X_0 \rightsquigarrow Y$ at (x^0, y^0) is introduced in [19] as follows. Recall that the epigraph of F is defined as the set

$$\operatorname{epi} F = \{(x, y) \in X_0 \times Y \mid x \in X_0,\ y \in F(x) + C\}\,.$$

The contingent cone $T(\operatorname{epi} F,\,(x^0, y^0))$ of the epigraph $\operatorname{epi} F$ at (x^0, y^0) is defined as the set of all $(x, y) \in X \times Y$ for which there exists a sequence $(x^n, y^n) \in \operatorname{epi} F$ and a sequence $\lambda_n > 0$ of positive reals such that $x = \lim_n \lambda_n (x^n - x^0)$ and $y = \lim_n \lambda_n (y^n - y^0)$. The contingent epiderivative of F at (x^0, y^0) is defined as a single-valued map $DF(x^0, y^0) : X \to Y$ whose epigraph equals the contingent cone to the epigraph of F at (x^0, y^0), i. e.

$$\operatorname{epi} DF(x^0, y^0) = T(\operatorname{epi} F,\,(x^0, y^0))\,.$$

Recall that the svf $F : X_0 \rightsquigarrow Y$ is said to be C-convex if X_0 is convex and $(1-t)F(x^0) + tF(x^1) \subset F((1-t)x^0 + tx^1) + C$ for all $x^0,\ x^1 \in X_0$ and all t satisfying $0 < t < 1$.

Theorem 6.1 in opposite to our Theorem 5.1 treats also problems with image space of infinite dimension. In the case of a finite dimensional image space and pointed closed convex cone C the established in 5.1 sufficient conditions are more general than those of Theorem 6.1. Obviously each C-convex svf $F : X_0 \rightsquigarrow Y$ is C-convex-along-rays at (x^0, y^0), where $x^0 \in X^0$, $y^0 \in F(x^0)$. The converse is not true as it is seen on the function F from Example 5.2. We have there

$$\frac{1}{2}\,F(x^0) + \frac{1}{2}\,F(x) \not\subset F(\,\frac{1}{2}\,x^0 + \frac{1}{2}\,x) \quad \text{for} \quad x \neq x^0\,.$$

If the contingent epiderivative $DF(x^0, y^0)$ exists, from its definition it follows that the vector $g_u = DF(x^0, y^0)(u)$ satisfies the inclusion $F(x^0 + tu) \subset y^0 + tg_u + C$ from Theorem 5.1. The existence of such a vector g_u for all $u \in X_0) \backslash \{0\}$ however does not imply the existence of the contingent epiderivative

$DF(x^0, y^0)$. For instance for the function F in Example 5.2 and $x^0 = 0$, $y^0 = (0,\, 0)$ $DF(x^0, y^0)$ does not exist. Indeed, assuming that the contingent epiderivative exists and putting $g_0 = DF(x^0, y^0)(0)$ we would have as above $F(0) \subset tg_0 + \mathbb{R}^2_+$ for all $t > 0$. However, from the definition of F it follows that such a vector does not exist.

After these remarks we can summarize, that while in Example 5.2 the pair (x^0, y^0) can be recognized as a w-minimizer on the base of Theorem 5.1, this cannot be done on the base of Theorem 6.1. The latter cannot be applied, since neither F is C-convex nor F is contingent epidifferentiable at (x^0, y^0).

Finally, let us recall, that in vector and set-valued optimization another concept of optimality exists, called by Luc [25] ideal efficient point (Jahn, Rauh [19] call it strong minimizer). Ideal efficient points exist rarely and for this reason usually are not put in the center of the theory. However, if a problem possesses a ideal solution, it is most desirable to be found. Ideal solution in multi-criterial optimization means a solution which supplies the optimum with respect to any single criterion. Looking closely at the structure of the ideal solutions, we will find in fact a scale of ideal solutions, closely related in their definitions to the points of efficiency defined in this paper. In particular we can introduce w-ideal, e-ideal, p-ideal and i-ideal solutions. The relations between w-minimizers, e-minimizers, p-minimizers and i-minimizers established in this paper obviously can be carried over ideal solutions.

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