Giovanni P. Crespi, Ivan Ginchev,
Matteo Rocca

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2004/22

UNIVERSITÀ DELL'INSUBRIA
FACOLTÀ DI ECONOMIA

http://eco.uninsubria.it

Printed in Italy in July 2004
Università degli Studi dell'Insubria
Via Ravasi 2, 21100 Varese, Italy

Two approaches toward constrained vector optimization and identity of the solutions

Giovanni P. Crespi
Université de la Vallé d'Aoste
Facoltà di Scienze Economiche
11100 Aosta, Italy
g.crespi@univda.it

Ivan Ginchev
Technical University of Varna
Department of Mathematics
9010 Varna, Bulgaria
ginchev@ms3.tu-varna.acad.bg

Matteo Rocca
University of Insubria
Department of Economics
21100 Varese, Italy
mrocca@eco.uninsubria.it

Abstract

In this paper we deal with a Fritz John type constrained vector optimization problem. In spite that there are many concepts of solutions for an unconstrained vector optimization problem, we show the possibility "to double" the number of concepts when a constrained problem is considered. In particular we introduce sense I and sense II isolated minimizers, properly efficient points, efficient points and weakly efficient points. As a motivation leading to these concepts we give some results concerning optimality conditions in constrained vector optimization and stability properties of isolated minimizers and properly efficient points. Our main investigation and results concern relations between sense I and sense II concepts. These relations are proved mostly under convexity type conditions.

2000 Mathematics Subject Classification: 90C29.

Keywords: Constrained vector optimization, Optimality conditions, Stability, Type of solutions and their identity, Vector optimization and convexity type conditions.

1 Introduction

In this paper we deal with a Fritz John type constrained vector optimization problem. In spite that there are many concepts of solutions for an unconstrained vector optimization problem, we show the possibility "to double" the number of concepts when a constrained problem is considered. In particular we introduce sense I and sense II isolated minimizers, properly efficient points, efficient points and weakly efficient points. In Section 2 as a motivation leading to these concepts we give some results concerning optimality conditions in constrained vector optimization and stability properties of isolated minimizers and properly efficient points. On these results we show when a preference to the one of the two type of concepts can be given. For instance, sense I concept are related to optimality conditions patterned on the traditionally results in constrained optimization, while sense II concepts are preferable when one is interested on the stability of the investigated problem (sense II concepts show stability both with respect to the objective and constraint data, while sense I concepts show stability only with respect to the objective data). These observations seem to show that of some importance is the question, when sense I and sense II concept coincide. In Section 3 we investigate this problem for isolated minimizers, in Section 4 for properly efficient points and in Section 5 for efficient and weakly efficient points.

2 Preliminaries

We consider the constrained vector optimization problem

$$\min\nolimits_C f(x)\,, \quad g(x) \in -K\,, \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^m$, $g : \mathbb{R}^n \to \mathbb{R}^p$. Here n, m and p are positive integers, $K \subset \mathbb{R}^p$ is a closed convex cone and we assume that a partial ordering on $\mathbb{R}^m$ is induced by a cone C which is closed and convex. The functions f and g are in general nonsmooth with domain $\mathbb{R}^n$. The latter is taken for simplicity, but the considerations can be extended straightforward to functions defined on an open set in $\mathbb{R}^n$. When some regularity is assumed , it will be explicitly said. For instance, Theorem 1 below is formulated for $C^{0,1}$ functions or in other words for locally Lipschitz functions.

Turn attention that the unconstrained problem

$$\min_C f(x) \tag{2}$$

is a particular case of problem (1). Below we give several solution concepts for problem (1). Each solution x^0 is supposed to be a feasible point, that is satisfying $x^0 \in g^{-1}(-K)$. This condition does not occur, when the definitions are adjusted for the unconstrained problem (2).

Recall, that the point x^0 is a weakly efficient, efficient, or strongly efficient point (in the paper we call it for brevity respectively w-minimizer, e-minimizer and strong e-minimizer) for the constrained problem (1) if there is a neighbourhood U of x^0 such that $f(x) - f(x^0) \notin -\operatorname{int} C$ for $x \in U \cap g^{-1}(-K)$ (respectively $f(x) - f(x^0) \notin (C \setminus \{0\})$ for $x \in U \cap g^{-1}(-K)$ or $f(x) - f(x^0) \notin -C$ for $x \in (U \setminus \{x^0\}) \cap g^{-1}(-K)$).

Given a cone $M \subset \mathbb{R}^k$ we define its positive polar cone by $M' = \{\xi \in \mathbb{R}^k \mid \langle \xi, x \rangle \geq 0\}$. For M closed convex it is known (see, Rockafellar [25, Chapter III, § 15]) that $M'' := (M')' = M$. We apply here positive polar cones for M equals C or K. Further for $x^0 \in K$ we use also the cone $K[g(x^0)]$ defined as follows. We put $K'[g(x^0)] = \{\xi \in K' \mid \langle \xi, g(x^0) \rangle = 0\}$ and define $K[g(x^0)] = (K'[g(x^0)])'$.

The oriented distance (see e.g. [15]) $D(y, M)$ from a point y to a set M in a metric space is defined by $D(y, M) = \operatorname{dist}(y, M) - \operatorname{dist}(y, M^c)$, where M^c stands for the complement of M. If M is a convex set in $\mathbb{R}^k$ it is shown in Ginchev, Hoffmann [12] that the oriented distance is expressed by $D(y, M) = \max_{\|\xi\|=1} \left(\inf_{a \in M} \langle \xi, a \rangle - \langle \xi, y \rangle\right)$, which in the case when M is a convex cone gives $D(y, -M) = \max_{\|\xi\|=1, \xi \in M'} \left(\langle \xi, y \rangle\right)$.

We recall here that the oriented distance function can be used to give scalar characterizations of vector optimality concepts [26].

Proposition 1. *Define*

$$\varphi(x) = \max\{\langle \xi, f(x) - f(x^0) \rangle | \xi \in C', \|\xi\| = 1\} \tag{3}$$

i) The feasible point $x^0 \in \mathbb{R}^n$ is a w-minimizer for problem (1), if and only if x^0 is a minimizer for the scalar problem

$$\min \varphi(x), \quad g(x) \in -K\,. \tag{4}$$

ii) The feasible point x^0 is a strong e-minimizer of problem (1) if and only if x^0 is a strong minimizer of problem (4), i.e. if and only if there exists a neighborhood U of x^0, such that $\varphi(x) - \varphi(x^0) > 0$ for all $x \in (U \setminus \{x^0\}) \cap g^{-1}(-K)$.

A concept of great importance in vector optimization is that of a properly efficient point (p-minimizer, for short). Usually the concept of a p-minimizers is defined in the references for the case of a pointed closed convex cone C. Then x^0 is said to be a p-minimizer for the constrained problem (2) if there is a pointed closed convex cone $\tilde{C}$ such that x^0 is a w-minimizer for the constrained problem $\min_{\tilde{C}} f(x)$. To overcome the assumption C pointed, in [11] the concept of a properly efficient point is defined as follows. For given $a > 0$ we define the closed cone

$$C(a) = \{y \in \mathbb{R}^m \mid D(y, C) \leq a\,\|y\|\}\,.$$

For the constrained problem (1) we say that the feasible point x^0 is a p-minimizer if $f(x) - f(x^0) \notin -\mathrm{int}\, C(a)$ for $x \in U \cap g^{-1}(-K)$. This definition can be rephrased saying that x^0 is a p-minimizer for problem (1) if there exists a closed cone $\tilde{C}$, such that $f(x) - f(x^0) \notin -\mathrm{int}\, \tilde{C}$ for $x \in g^{-1}(-K)$ (observe that here the convexity requirement on $\tilde{C}$ is dropped).

The concept of a properly efficient point has been introduced into multicriteria optimization by Kuhn and Tucker [20] as solutions being stable with respect to preference, in the sense that they exclude a situation, when a first-order gain in one criterion can be obtained for only higher-order loss in another criterion. While the Kuhn-Tucker concept of a properly efficient point essentially makes an use of certain type of necessary optimality conditions and constraint qualifications, this dependence is overcome in the definition given by Geoffrion [7]. Still, Geoffrion's notion of proper efficiency uses essentially the coordinate character of the criteria and does not admit a straightforward generalization to vector optimization with respect to more general partial order. This limitation does not occur in Borwein [6] who defines proper efficiency with respect to order given by cones. The notion of proper efficiency has undergone some historical development and nowadays it does not exist a unique commonly accepted definition of a properly efficient point. Recently the most often used understanding of this notion is probably the one proposed in Henig [14]. Survey on proper efficiency and different approaches to this concept one can find e.g in Podinovskiy, Nogin [24] or Guerraggio, Molho, Zaffaroni [13].
Finally, we introduce the notion of isolated minimizer of order k for the constained problem 1. For scalar problems this concept has been given by Auslender [1]. This concept is generalized and studied for vector problems by Ginchev [8] and Ginchev, Guerraggio, Rocca [9, 10], and independently by Jiménez [17] and Jiménez, Novo [18].

We say that the feasible point x^0 is an isolated minimizer of order $\kappa \geq 1$ for (1) if there is a constant $A > 0$ and a neighbourhood U of x^0 such that $D(f(x) - f(x^0), -C) \geq A\, \|x - x^0\|^\kappa$ for $x \in U \cap g^{-1}(-K)$.
Given a point x^0, together with constrained problem (1) we consider the unconstrained problem

$$\min_{C \times K[g(x^0)]} \left(f(x),\, g(x)\right). \tag{5}$$

We will say that the feasible (for problem (1)) point x^0 is a w-minimizer (respectively e-minimizer, strong e-minimizer, p-minimizer, isolated minimizer of order κ) in sense II for problem (1) when it is a w-minimizer (respectively e-minimizer, strong e-minimizer, p-minimizer, isolated minimizer of order κ) for problem (5). We will refer further to the defined earlier classical solution concepts for problem (1) as to the sense I concepts. If like in Theorem 1 we do not mention explicitly of which sense are the applied minimizer, we will accept by default that they are minimizers in sense I.

A motivation to associate to the constrained problem (1) the unconstrained problem (5) is the coincidence of the sense I and sense II isolated minimizers, which is explained at the end of Section 3. This coincidence is obtained on the base of some first-order optimality conditions given below in Theorem 1.

As some prerequisite we need to introduce the concept of a Dini derivative.

Given a $C^{0,1}$ function $\Phi : \mathbb{R}^n \to \mathbb{R}^k$ we define the Dini directional derivative (we use to say just Dini derivative) $\Phi'_u(x^0)$ of Φ at x^0 in direction $u \in \mathbb{R}^n$ as the set of the cluster points of $(1/t)(\Phi(x^0 + tu) - \Phi(x^0))$ as $t \to +0$, that is as the Kuratowski limit

$$\Phi'_u(x^0) = \operatorname*{Limsup}_{t \to +0} \frac{1}{t} \left(\Phi(x^0 + tu) - \Phi(x^0)\right).$$

If Φ is Fréchet differentiable at x^0 then the Dini derivative is a singleton, coincides with the usual directional derivative and can be expressed in terms of the Fréchet derivative $\Phi'(x^0)$ (called sometimes the Jacobian of Φ at x^0) by $\Phi'_u(x^0) = \Phi'(x^0)u$.
In connection with problem (1) we deal with the Dini directional derivative of the function $\Phi : \mathbb{R}^n \to \mathbb{R}^{m+p}$, $\Phi(x) = (f(x),\, g(x))$, and then we use to write $\Phi'_u(x^0) = (f(x^0),\, g(x^0))'_u$. If at least one of

the derivatives $f'_u(x^0)$ and $g'_u(x^0)$ is a singleton, then $(f(x^0), g(x^0))'_u = (f'_u(x^0), g'_u(x^0))$. Let us turn attention that always $(f(x^0),\, g(x^0))'_u \subset f'_u(x^0) \times g'_u(x^0)$, but in general these two sets do not coincide. Indeed, for any $C^{0,1}$ function f, $(f(x^0),\, f(x^0))'_u$ is the diagonal of $f'_u(x^0) \times f'_u(x^0)$. If $f'_u(x^0)$ is not a singleton, then the two sets are different.

Theorem 1 (First-order conditions, [10]). *Let f, g be $C^{0,1}$ functions and consider problem (1).*

(Necessary Conditions) *Let x^0 be a w-minimizer of problem (1). Then for each $u \in \mathbb{R}^n$ the following condition is satisfied:*

$$\mathbb{N}'_{0,1}: \quad \begin{array}{c} \forall (y^0,\, z^0) \in (f(x^0),\, g(x^0))'_u : \exists (\xi^0,\, \eta^0) \in C' \times K' : \\ (\xi^0,\, \eta^0) \neq (0,\, 0), \quad \langle \eta^0,\, g(x^0) \rangle = 0 \quad \textit{and} \quad \langle \xi^0,\, y^0 \rangle + \langle \eta^0,\, z^0 \rangle \geq 0\,. \end{array}$$

(Sufficient Conditions) *Let $x^0 \in \mathbb{R}^n$ and suppose that for each $u \in \mathbb{R}^n \setminus \{0\}$ the following condition is satisfied:*

$$\mathbb{S}'_{0,1}: \quad \begin{array}{c} \forall (y^0,\, z^0) \in (f(x^0),\, g(x^0))'_u : \exists (\xi^0,\, \eta^0) \in C' \times K' : \\ (\xi^0,\, \eta^0) \neq (0,\, 0), \quad \langle \eta^0,\, g(x^0) \rangle = 0 \quad \textit{and} \quad \langle \xi^0,\, y^0 \rangle + \langle \eta^0,\, z^0 \rangle > 0\,. \end{array}$$

Then x^0 is an isolated minimizer of first order for problem (1).

Conversely, if x^0 is an isolated minimizer of first order for problem (1) and the constraint qualification $\mathbb{Q}_{0,1}(x^0)$ (see below) holds, then condition $\mathbb{S}'_{0,1}$ is satisfied.

In the Sufficient Conditions part of Theorem 1 the following constraint qualification appears, which generalizes to $C^{0,1}$ functions the Kuhn-Tucker constraint qualification [20] (compare also with Mangasarian [22, p. 102]):

$$\mathbb{Q}_{0,1}(x^0)\ : \quad \begin{array}{c} \text{If } g(x^0) \in -K \text{ and } \dfrac{1}{t_k}\left(g(x^0 + t_k u^0) - g(x^0)\right) \to z^0 \in -K[g(x^0)] \\ \text{then } \exists\, u^k \to u^0 : \exists\, k_0 \in \mathbb{N} : \forall\, k > k_0 : g(x^0 + t_k u^k) \in -K\,. \end{array}$$

A motivation to introduce sense II concepts give also the stability properties obeyed by the p-minimizers and the isolated minimizers.

The stability properties obeyed by the properly efficient points have been a subject of investigation since short after the notion appeared in the literature, see e.g. Benson, Morin [5]. Stability can be understood in different ways as one sees in Miglierina, Molho [23]. Their approach concerns however efficient boundaries of sets and it is not appropriate for comparison of different notions of proper efficiency and stability for constrained problems. Some peculiarities concerning stability when constrained optimization problems are investigated consider Balayadi, Sonntag, Zălinescu [2], but their approach relates to usual and not to vector optimization. Stability properties obey also the isolated minimizers. For scalar problems this has been shown in Auslender [1]. For vector optimization this topic has been investigated in Ginchev, Guerraggio, Rocca [11]. Theorem 2 gives stability properties for properly efficient points and Theorem 3 for isolated minimizers.

Together with the constrained problem (1) we consider also the perturbed problem

$$\min_{\tilde{C}} \tilde{f}(x)\,, \quad \tilde{g}(x) \in -\tilde{K}\,. \tag{6}$$

Theorem 2 (Stability of p-minimizers, [11]). **a.** *Let x^0 be a p-minimizer in sense I for the constrained problem (1). Then there exists $\delta > 0$, such that for the perturbed problem (6) with $\tilde{C} \subset C(\delta)$, $\tilde{K} = K$, $\tilde{f} = f$, $\tilde{g} = g$, the point x^0 is also a p-minimizer in sense I.*

b. *Let x^0 be a p-minimizer in sense II for the constrained problem (1). Then there exists $\delta > 0$, such that if $\tilde{C} \subset C(\delta)$, $\tilde{K} \subset K[g(x^0)](\delta)$, $\tilde{f} = f$, $\tilde{g} = g$, then the point x^0 is a p-minimizer for the problem $\min_{\tilde{C} \times \tilde{K}}(\tilde{f}(x),\, \tilde{g}(x))$.*

Theorem 3 (Stability of isolated minimizers, [11]). **a.** *Let x^0 be an isolated minimizer of first order in sense I for the constrained problem (1) with f and g locally Lipschitz functions. Then there exists $\delta > 0$ and a neighbourhood U of x^0, such that for the perturbed problem (6) with $\tilde{C} \subset C(\delta)$, $\tilde{K} = K$, $\|\tilde{f}(x) - f(x)\| \leq \delta\|x - x^0\|$ for $x \in U$, $\tilde{g} = g$, the point x^0 is also an isolated minimizer of first order in sense I.*

b. *Let x^0 be an isolated minimizer of first order in sense II for the constrained problem (1) with f and g locally Lipschitz functions. Then there exists $\delta > 0$ and a neighbourhood U of x^0, such that if $\tilde{C} \subset C(\delta)$, $\tilde{K} \subset K[g(x^0)](\delta)$, $\|\tilde{f}(x) - f(x)\| \leq \delta\|x - x^0\|$ for $x \in U$, $\|\tilde{g}(x) - g(x)\| \leq \delta\|x - x^0\|$ for $x \in U$, then the point x^0 is an isolated minimizer of first order for the problem $\min_{\tilde{C}\times\tilde{K}}(\tilde{f}(x),\, \tilde{g}(x))$.*

Theorems 2 and 3 show that p-minimizers are stable under perturbations of the ordering cones, while isolated minimizers are stable under perturbations of both the cones and the given functions. They show also, that sense I concepts are stable under perturbations of the objective data, while sense II concepts are stable under perturbations of both the objective and constrained data. From this point of view sense II concepts are advantageous, since it is preferable to deal with a problem, which is stable with respect to all data, than with one, which is stable with only part of the data.

On the other hand, when dealing with the optimality conditions expressed in Theorem 1, we see certain advantage of the sense I concepts, based on the fact that the derived optimality conditions show certain similarity with known classical optimality conditions. Obviously, if one wishes to combine the advantages of both sense I and sense II concepts, a natural task is to establish their relations, and in particular to find conditions, under which they coincide.

There exist relations between different type of solutions, for instance obviously the e-minimizers are w-minimizers. Relations between p-minimizers, isolated minimizers of first order and the so called strict minimizers are investigated in [4] and [11], and between more special optimality notions in [26]. In the present paper our task is bit different. Dealing with a particular optimality notion among the mentioned isolated minimizers of first order, p-minimizers, e-minimizers and w-minimizers we establish the relations of the respective sense I and sense II concepts. As we will see, several such results hold under convexity type conditions on f and g. In Section 3 we consider isolated minimizers, in Section 4 p-minimizers, and Section 5 is devoted to w-minimizers and e-minimizers.

3 Isolated minimizers

The next proposition states that the isolated minimizers in sense II are a subset of the isolated minimizers in sense I.

Proposition 2. *If the point x^0, feasible for problem (1), is an isolated minimizer of order κ in sense II, then it is also an isolated minimizer of order κ in sense I.*

Proof. Since x^0 is an isolated minimizer of order κ in sense II, then there exists a positive constant A, such that for x in a suitable neighborhood U of x^0 it holds

$$\max_{(\xi,\gamma)\in(C\times K[g(x^0)])'\cap S} \langle(\xi,\gamma),(f(x)-f(x^0),g(x)-g(x^0))\rangle \geq A\|x-x^0\|^{\kappa}.$$

Here S stands for the unit sphere in $\mathbb{R}^m \times \mathbb{R}^p$. Since $(C \times K[g(x^0)])' = C' \times K'[g(x^0)]$, we obtain that for every $x \in U$, there exists $\xi_x \in C'$, $\gamma_x \in K'[g(x^0)]$, with $(\xi_x, \gamma_x) \neq (0, 0)$, such that

$$\langle\xi_x, f(x) - f(x^0)\rangle + \langle\gamma_x, g(x) - g(x^0)\rangle \geq A\|x - x^0\|^{\kappa}$$

and hence, for every feasible $x \in U$ we have $\langle\gamma_x, g(x) - g(x^0)\rangle \leq 0$. So we get

$$\langle\xi_x, f(x) - f(x^0)\rangle \geq A\|x - x^0\|^{\kappa}.$$

For every feasible $x \in U$, we have $\xi_x \neq 0$. In fact, if $\xi_x = 0$, we must have $\gamma_x \neq 0$ and we get the absurdo

$$\langle \gamma_x, g(x) - g(x^0) \rangle \geq A\|x - x^0\|^\kappa > 0.$$

Since $(\xi, \gamma) \in S$, we have $0 < \sup_{x \in U \cap g^{-1}(-K)} \|\xi_x\| = \tau < +\infty$ and hence

$$\langle \frac{\xi_x}{\|\xi_x\|}, f(x) - f(x^0) \rangle \geq \frac{A}{\|\xi_x\|}\|x - x^0\|^\kappa \geq \frac{A}{\tau}\|x - x^0\|^\kappa$$

and this proves that x^0 is an isolated minimizer of order κ in sense I. □

The previous result is not reversible, as shown by the following example.

Example 1. *Consider the constrained problem (1), with $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}^2$, $f(x) = e^{-1/|x|}$, $g(x) = (e^{-1/|x|}, -x)$ for $x \neq 0$, $f(0) = 0$, $g(0) = (0, 0)$ and $K = \mathbb{R}^2_+$. Then, $x^0 = 0$ is the unique feasible point and hence is an isolated minimizer of every order , in sense I, but it is easily seen that x^0 is not an isolated minimizer of any order in sense II.*

In the next two sections we assume local convexity properties at x^0 of the functions involved in the considered problem (1) to guarantee the coincidence of the sense I and sense II concepts. Since in the previous example both f and g enjoy such local convexity properties at x^0, we see, that the local convexity assumptions are not relevant to revert Proposition 2. The next result gives a reversal of Proposition 2, for the case of isolated minimizers of order 1, under a constraint qualification condition, as an application of Theorem 1.

Proposition 3 (Ginchev, Guerraggio, Rocca [11]). *Let f and g be locally Lipschitz functions. If x^0 is an isolated minimizer of first order in sense I, for the constrained problem (1) and the constraint qualification $\mathbb{Q}_{0,1}(x^0)$ holds, then x^0 is an isolated minimizer of first order in sense II.*

Proof Since x^0 is an isolated minimizer of first order and the constraint qualification $\mathbb{Q}_{0,1}(x^0)$ is satisfied, then the reversal of the Sufficient Conditions in Theorem 1 gives that condition $\mathbb{S}'_{0,1}$ holds. These conditions are however identical with the sufficient conditions, which one obtains, when reformulating Theorem 1 for the unconstrained problem (5) and in consequence x^0 is an isolated minimizer of first order for problem (5). □

The above theorem clarifies the motivations, which led us to associate to the constrained problem (1) the unconstrained problem (5). Namely, the coincidence of the optimality conditions for the problems (1) and (5). In fact, the only difference we get, is that for the unconstrained problem (5) the reversal of the sufficient conditions is not restricted by the appearance of constraint qualifications. The coincidence of the optimality conditions gives the coincidence of the isolated minimizers of first order when the constraint qualification is satisfied. This inspires us similarly to consider each concept of efficiency in two senses and to distinguish conditions, under which sense I and sense II concepts coincide.

4 Proper efficiency and convexity

In Section 2, we recalled the concept of p-minimizer in sense II and the stability properties of this notion, and as well those of the classical notion of p-minimizer (i.e. p-minimizers in sense I). In this section we explore the links between these two notion under convexity assumptions on the constraint function. We will see that, under convexity of the constraint function g and radial continuity of f, p-minimizers in sense II are a subset of p-minimizers in sense I.
We need to recall the following definitions (see e.g. [16, 21]).

Definition 1. *Let $D \subset \mathbb{R}^p$ be a convex cone.*

i) *The function* $g : \mathbb{R}^n \to \mathbb{R}^p$ *is said to be locally* D*-convex at* x^0 *when there exists a neighbourhood* U *of* x^0*, such that*

$$g((1-t)x' + tx) \in (1-t)g(x') + tg(x) - D$$

for every $x', x \in U$ *and* $t \in (0,1)$*.*

ii) *The function* $g : \mathbb{R}^n \to \mathbb{R}^p$ *is said to be locally* D*-quasiconvex at* x^0 *when there exists a neighbourhood* U *of* x^0*, such that for every* $x', x \in U$ *and for every* $t \in (0,1)$ *it holds*

$$f(x) - f(x') \in -D \Rightarrow f((1-t)x' + tx) - f(x') \in -D\,.$$

Clearly, every locally D-convex function is locally D-quasiconvex.
We recall furthermore that a function $g : \mathbb{R}^n \to \mathbb{R}^p$ is said to be radially continuous at x^0 when its restriction along rays starting at x^0 is continuous. In the sequel we will assume that C and K are cones with nonempty interior.

Proposition 4. *Let* $f : \mathbb{R}^n \to \mathbb{R}^m$ *be radially continuous at the point* x^0*, feasible for problem (1), and let* g *be locally* $\operatorname{int} K[g(x^0)]$*-convex at* x^0*. If* x^0 *is a* p*-minimizer in sense II, then it is also a* p*-minimizer in sense I.*

Proof. Let the feasible point x^0 be a p-minimizer in sense II for problem (1). Then there exist closed cones $\tilde{C}$ and $\tilde{K}$, with $(C\backslash\{0\} \times K[g(x^0)]\backslash\{0\}) \subset (\operatorname{int}\tilde{C} \times \operatorname{int}\tilde{K})$ and a neighborhood U of x^0, such that $(f(x) - f(x^0), g(x) - g(x^0)) \notin (-\operatorname{int}\tilde{C}) \times (-\operatorname{int}\tilde{K})$, for every $x \in U$. Ab absurdo, assume that x^0 is not p-minimizer in sense I. Hence one can find a point $x \in U \cap g^{-1}(-K)$, such that $f(x) - f(x^0) \in -\operatorname{int}\tilde{C}$. If $K[g(x^0)] = \mathbb{R}^p$, a contradiction is achieved. Assume now that $K[g(x^0)] \neq \mathbb{R}^p$. Then, since $g(x) \in -K \subset K[g(x^0)]$, for every $\xi \in K'[g(x^0)]$, we have $\langle \xi, g(x) - g(x^0)\rangle = \langle \xi, g(x)\rangle \leq 0$. Hence, $g(x) - g(x^0) \in -K[g(x^0)]$. Without loss of generality we can assume that g is locally $\operatorname{int} K[g(x^0)]$-convex at x^0 on the neighborhood U. The local convexity assumption now gives

$$g((1-t)x^0 + tx) - g(x^0) \in t(g(x) - g(x^0)) - \operatorname{int} K[g(x^0)],$$

for every $t \in (0,1)$. Hence

$$g((1-t)x^0 + tx) - g(x^0) = t(g(x) - g(x^0)) - \alpha(x^0)\,,$$

where $\alpha(x^0) \in \operatorname{int} K[g(x^0)]$. So, for every $\xi \in K'[g(x^0)]$, $\xi \neq 0$, we obtain

$$\langle \xi, g((1-t)x^0 + tx) - g(x^0)\rangle < 0\,,$$

that is $g((1-t)x^0 + tx) - g(x^0) \in -\operatorname{int} K[g(x^0)] \subset -\operatorname{int}\tilde{K}$. Since f is radially continuous at x^0 and $-\operatorname{int}\tilde{C}$ is open, then, for t near enough to 1, we obtain $f((1-t)x^0 + tx) - f(x^0) \in -\operatorname{int}\tilde{C}$ and $g((1-t)x^0 + tx) - g(x^0) \in -\operatorname{int}\tilde{K}$, which is a contradiction. $\square$

The previous proposition cannot be reverted, as the following example shows.

Example 2. *Let* $f : \mathbb{R} \to \mathbb{R}^2$ *and* $g : \mathbb{R} \to \mathbb{R}$ *be defined as* $f(x) = (-x, x^2)$, $g(x) = x^2$ *and* $C = \mathbb{R}^2_+$, $K = \mathbb{R}_+$. *Here* f *and* g *satisfy the assumptions of the previous proposition and* $x_0 = 0$ *is* p*-minimizer in sense I for problem (1). Anyway,* x_0 *is not a* p*-minimizer in sense II.*

Indeed, observe that x_0 is the unique feasible point and hence is a p-minimizer in sense I. Anyway we have $K[g(x_0)] = K$, $(f(x), g(x)) = (-x, x^2, x^2)$ and it is easily seen that x_0 is not p-minimizer in sense II.

Remark 1. *Observe that, from* $K \subset K[g(x^0)]$ *it follows* $\operatorname{int} K \subset \operatorname{int} K[g(x^0)]$ *and hence local* $\operatorname{int} K$*-convexity at* x^0 *implies local* $\operatorname{int} K[g(x^0)]$*-convexity at* x^0*. Hence the* $\operatorname{int} K[g(x^0)]$*-convexity assumption in the previous result, can be replaced by the stronger (and more common)* $\operatorname{int} K$*-convexity assumption. Analogous observation holds for the results of Section 5.*

5 Efficient points and weakly efficient points

In principle, as already said in Section 2, each type of minimizer of problem (1) admits the two considered approaches (i.e. can be considered in sense I and in sense II). In this section we consider w-minimizers and e-minimizers. We investigate the links between these sense I and sense II notions under (generalized) convexity assumptions.

Proposition 5. *Let x^0 be a feasible point for problem (1). Assume f is radially continuous at the point x^0 and g is locally $\operatorname{int} K[g(x^0)]$-convex at x^0. If x^0 is a w-minimizer in sense II, then x^0 is a w-minimizer in sense I for problem (1).*

Proof. It is similar to that of Proposition 4 and is omitted. □

When $K = \mathbb{R}^p_+$, it is easily seen that $K[g(x^0)] = \{(z_1, \dots, z_p) \in \mathbb{R}^p |\, z_i \geq 0, i \in I(x^0)\}$, where $I(x^0) = \{i = 1, \dots, p |\, g_i(x^0) = 0\}$. Let $g = (g_1, \dots, g_p)$ and $g^{I(x^0)} = (g_i,\, i \in I(x^0))$. Hence problem (5) can be rephrased as

$$\min_{C \times \mathbb{R}^{I(x^0)}_+} (f, g^{I(x^0)}) , \quad x \in \mathbb{R}^n . \tag{7}$$

Next corollary is a rephrasing of Proposition 5 in the case $K = \mathbb{R}^p_+$.

Corollary 1. *Let $K = \mathbb{R}^p_+$ and assume that f is radially continuous at the point x^0 feasible for problem (1) and that the functions g_i, $i \in I(x^0)$, are locally strictly convex at x^0. If x^0 is a w-minimizer for problem (7) (i.e. a w-minimizer in sense II), then x^0 is a w-minimizer in sense I.*

Proposition 5 in general is not reversible, as the following example shows.

Example 3. *Consider functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}^3$ defined as $f(x) = -x^3$ and $g(x) = (x^2, x^2 - 1, x - 1)$. Let $C = \mathbb{R}_+$ and $K = \{(z_1, z_2, z_3) \in \mathbb{R}^3 |\, z_3 \leq 0,\, z_3^2 \geq z_1^2 + z_2^2\}$. Then $x_0 = 0$ is w-minimizer in sense I, but not in sense II.*

Indeed, $-K[g(x_0)]$ is a halfspace determined by the unique tangent plane to the cone $-K$ at $g(x_0)$. More precisely, $-K[g(x_0)] = \{(z_1, z_2, z_3) \in \mathbb{R}^3 |\, z_3 \geq z_2\}$. It is easily seen that g is locally $\operatorname{int} K[g(x_0)]$-convex at x_0 and that the points $x \in \mathbb{R}$ with $|x|$ small enough, are feasible if and only if $x < 0$. Hence, obviously x_0 is w-minimizer in sense I. But for $x > 0$ small enough, we have $g(x) - g(x_0) = (x^2, x^2, x) \in -\operatorname{int} K[g(x_0)]$ and $f(x) - f(x_0) < 0$. Hence x_0 is not w-minimizer in sense II.

When $K = \mathbb{R}^p_+$, we can obtain some reversal of Corollary 1.

Proposition 6. *Let $K = \mathbb{R}^p_+$ and let g_i, $i \notin I(x^0)$ be continuous at x^0. If x^0 is a w-minimizer in sense I, then it is also a w-minimizer in sense II.*

Proof. If x^0 is a w-minimizer in sense I, then there exists a neighborhood U of x^0 such that $\forall x \in U \cap g^{-1}(-K)$, $f(x) - f(x^0) \notin -\operatorname{int} C$. Assume, ab absurdo, that x^0 is not a w-minimizer in sense II. Hence, for every neighborhood U of x^0 there exists a point $x \in U$, such that

$$(f(x) - f(x^0),\, g^{I(x^0)}(x)) \in -\operatorname{int} (C \times \mathbb{R}^{I(x^0)}_+)$$

For $i \notin I(x^0)$, we have $g(x^0) < 0$ (since x^0 is feasible) and from the continuity assumption we get $g_i(x) < 0$ in a suitable neighborhood of x^0. This contradicts to x^0 w-minimizer in sense I. □

The previous proposition is not extendable to general cones K, as shown by Example 3.

A reversal of Proposition 5 in the case of general cone K can be obtained under convexity conditions also on the objective function f. We need some preliminary results.

Let $(\xi^0, \eta^0) \in (C' \times K'[g(x^0)]) \backslash \{(0,0)\}$. Consider the scalar function

$$\varphi^0(x) = \langle \xi^0, f(x) - f(x^0) \rangle + \langle \eta^0, g(x) \rangle , \quad x \in g^{-1}(-K) , \tag{8}$$

and let x^0 be a feasible point for problem (1). Clearly, if x^0 is a minimizer for the scalar function φ^0, then x^0 is a w-minimizer for problem (5) (i.e. a w-minimzer in sense II for problem (1)). Next proposition states that indeed (under regularity conditions) one gets that linearly scalarized solutions of problem (5) are w-minimizers in sense I for problem (1).

Proposition 7. *Let x^0 be a feasible point for problem (1). If x^0 is a minimizer for function φ^0 with $\xi^0 \neq 0$, then x^0 is a w-minimizer in sense I.*

Proof. We show, that the made assumptions imply that x^0 is a minimizer of the scalar function (3), whence according to Proposition 1 x^0 is w-minimizer for problem (1). Let U be the neighbourhood of x^0, for which $\varphi^0(x) \geq \varphi^0(x^0)$ for $x \in U \cap g^{-1}(-K)$. Without loss of generality, we may assume that $\|\xi^0\| = 1$, otherwise we replace in (8) ξ^0 by $\xi^0/\|\xi^0\|$. Fix $x \in U \cap g^{-1}(-K)$. Then for the function φ in (4) we have

$$\varphi(x) \geq \langle \xi^0,\, f(x) - f(x^0) \rangle \geq \langle \xi^0,\, f(x) - f(x^0) \rangle + \langle \eta^0,\, g(x) \rangle = \varphi^0(x) \geq \varphi^0(x^0) = 0 = \varphi(x^0) ,$$

which had to be demonstrated. Here we have applied that $\langle \eta^0,\, g(x) \rangle \leq 0$ coming from $g(x) \in -K$, and $\langle \eta^0,\, g(x^0) \rangle = 0$ coming from $\eta^0 \in K'[g(x^0)]$. □

Proposition 8. *Let x^0 be a feasible point for problem (1). Assume that for every neighbourhood U of x^0 there exists a point $x \in U \cap g^{-1}(-K)$, such that $\langle \eta, g(x) \rangle < 0$, for all $\eta \in K'[g(x^0)] \backslash \{0\}$. Then, if x^0 is a minimizer of the scalar function (8), for some $(\xi^0, \eta^0) \in C' \times K'[g(x^0)] \backslash \{(0,0)\}$, we have $\xi^0 \neq 0$.*

Proof. Ab absurdo, let $\xi^0 = 0$. We have $\varphi^0(x) \geq \varphi^0(x^0) = 0$, that is $\langle \eta^0, g(x) \rangle \geq 0$, which is a contradiction. □

Remark 2. *The assumption of the previous proposition can be regarded as a Slater-type constraint qualification (see e.g. [3]).*

Proposition 9. *Let x^0 be a feasible point for problem (1) and assume that $g^{-1}(-K)$ is not a singleton. If g is locally $\operatorname{int} K[g(x^0)]$-convex at x^0, then the Slater-type constraint qualification of the previous proposition holds.*

Proof. From the $\operatorname{int} K[g(x^0)]$-convexity of g, we obtain easily that there exists a neighbourhood U of x^0, such that $\forall \xi \in K'[g(x^0)] \backslash \{0\}$, $\forall t \in (0,1)$ and $\forall x \in U \cap g^{-1}(-K)$, it holds $\langle \xi, g((1-t)x^0 + tx) - g(x^0) \rangle < 0$, i.e. $\langle \xi, g((1-t)x^0 + tx) \rangle < 0$, which gives the desired conclusion. □

Proposition 10. *Let f be C-convex, g be $K[g(x^0)]$-convex and assume that the Slater-type constraint qualification of Proposition 8 holds. If the point x^0 is a w-minimizer in sense I, then it is also a w-minimizer in sense II.*

Proof. Since f is C-convex and x^0 is a w-minimizer in sense I, then it is known that there exists a vector $\xi^0 \in C'\backslash\{0\}$, such that x^0 is a minimizer for the scalar function $\langle \xi^0, f(x)\rangle$. Since this function is convex, from classical results in convex optimization we obtain the existence of a vector $\eta^0 \in K'[g(x^0)]$, and a scalar $\theta^0 \geq 0$, with $(\theta^0, \eta^0) \neq (0,0)$, such that x^0 is a minimizer for the function $\theta^0\langle \xi^0, f(x)\rangle + \langle \eta^0, g(x)\rangle$. Since the Slater-type constraint qualification holds,we can assume $\theta^0 = 1$, so that x^0 is a minimizer for the function $\langle \xi^0, f(x)\rangle + \langle \eta^0, g(x)\rangle$ and hence for function φ^0. The result now follows applying Proposition 7. □

Next propositions link e-minimizers and strong e-minimizers in sense I and II. We omit the easy proofs.

Proposition 11. *Let x^0 be a feasible point for problem (1) and assume that f is locally $C\backslash\{0\}$-quasiconvex at x^0 and that g is locally $\operatorname{int} K[g(x^0)]$-convex at x^0. Then, if x^0 is an e-minimizer in sense II, it is also an e-minimizer in sense I.*

Proposition 12. *Let x^0 be a feasible point for problem (1) and assume that f is locally C-quasiconvex at x^0 and that g is locally $K[g(x^0)]$-convex at x^0. Then, if x^0 is a strong e-minimizer in sense II, it is also a strong e-minimizer in sense I.*

Also the previous propositions are not reversible. In Example 3, f is C-quasiconvex (and also $C\backslash\{0\}$-quasiconvex), while g is $\operatorname{int} K[g(x^0)]$-convex. The point x^0 is a strong e-minimizer (and hence also e-minimizer) in sense I, but not in sense II.

In Section 3, Proposition 3, we could prove under suitable conditions (the constraint qualification $\mathbb{Q}_{0,1}(x^0)$) that the sense I isolated minimizers are sense II isolated minimizers. This proof is based on the existing reversal of the isolated minimizers part (Sufficient Conditions) of Theorem 1. We could expect, that similar statement for the sense I and sense II w-minimizers under suitable conditions would be implied by an eventual reversal of the w-minimizers part (Necessary Conditions) of Theorem 1. Next we propose such a reversal under convexity type conditions.

Theorem 4. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ be $C^{0,1}$ functions. and $C \subset \mathbb{R}^m$ and $K \subset \mathbb{R}^p$ be closed convex cones. Let $x^0 \in \mathbb{R}^n$. Suppose that at x^0 the function f is locally C-convex and g is locally $K[g(x^0)]$-convex. Assume that for every $u \in \mathbb{R}^m \setminus \{0\}$ and for every $(y^0,\, z^0) \in (f(x^0),\, g(x^0))'_u$ there exists a couple of vectors $(\xi^0,\, \eta^0) \in C' \times K'$ such that $\xi^0 \neq 0$, $\langle \eta^0,\, g(x^0)\rangle = 0$ and $\langle \xi^0,\, y^0\rangle + \langle \eta^0,\, z^0\rangle \geq 0$. Then x^0 is a w-minimizer (in sense I) for the constrained problem (1).*

Proof. Let f be locally C-convex in the neighbourhood U of x^0. Fix $x \in U \cap g^{-1}(-K)$, $x^0 \neq 0$. Then we get $f((1-t)x^0 + tx) \in (1-t)f(x^0) + tf(x) - C$ for $0 < t < 1$, whence

$$\frac{1}{t}\,\left(f(x^0 + t(x - x^0)) - f(x^0)\right) \in f(x) - f(x^0) - C\,.$$

Similarly, applying that that g is $K[g(x^0)]$-convex in U (without loss of generality U can be chosen common for f and g), we get

$$\frac{1}{t}\,\left(g(x^0 + t(x - x^0)) - g(x^0)\right) \in g(x) - g(x^0) - K[g(x^0)]\,.$$

Now let $(y^0,\, z^0) \in (f(x^0),\, g(x^0))'_{x-x^0}$ (the existence of at least one such pair $(y^0,\, z^0)$ follows from the $C^{0,1}$ property of f and g, see [10]). From the above inclusions with account of the closedness of the cones C and $K[g(x^0)]$ we get $y^0 \in f(x) - f(x^0) - C$ and $z^0 \in g(x) - g(x^0) - K[g(x^0)]$. Choose now $(\xi^0,\, \eta^0) \in C' \times K'[g(x^0)]$ according to the hypotheses such that $\langle \xi^0,\, y^0\rangle + \langle \eta^0,\, z^0\rangle \geq 0$ (turn attention that $\eta^0 \in K'$, $\langle \eta^0,\, g(x^0)\rangle = 0$, is equivalent to $\eta^0 \in K'[g(x^0)]$). We have $\langle \eta^0,\, g(x^0)\rangle = 0$ and $\langle \eta^0,\, g(x)\rangle \leq 0$, the latter is a consequence of $g(x) \in -K$. Therefore

$$\langle \xi^0,\; f(x) - f(x^0)\rangle \geq \langle \xi^0,\; f(x) - f(x^0)\rangle + \langle \eta^0,\; g(x) - g(x^0)\rangle$$

$$= \langle \xi^0,\ y^0 + c\rangle + \langle \eta^0,\ z^0 + c_1\rangle \geq \langle \xi^0,\, y^0\rangle + \langle \eta^0,\, z^0\rangle \geq 0\,.$$

Here $c \in C$ and $c_1 \in K[g(x^0]$. Thus $\langle \xi^0,\ f(x) - f(x^0)\rangle \geq 0$. Since $\xi^0 \in C'$ and $\xi^0 \neq 0$, we get $f(x) - f(x^0) \notin -\mathrm{int}\, C$, that is x^0 is a w-minimizer for the constrained problem (1). $\square$

Now we illustrate an application of Theorem 4.

Proposition 13. *Let f and g be C^1 functions. Let x^0 be a w-minimizer in sense II for the constrained problem (1). Suppose that at x^0 the function f is locally C-convex and g is locally $K[g(x^0)]$-convex. Assume that the Slater-type constraint qualification from Proposition 8 is satisfied. Then x^0 is also a w-minimizer in sense I for (1).*

Proof. Since x^0 is a w-minimizer for (5), then from the Necessary Conditions of Theorem 1 applied for (5) and adjusted for C^1 problems, we see that there exists a pair $(\xi^0,\, \eta^0) \in C' \times K'[g(x^0)]$, $(\xi^0,\, \eta^0) \neq (0,\, 0)$, such that for all $u \in \mathbb{R}^n$ it holds $\langle \xi^0,\, f'(x^0)u\rangle + \langle \eta^0,\, g'(x^0)u\rangle \geq 0$. Saying "adjusted", we mean some replacement of the conditions. (It has been explained in [10], that passing from $C^{0,1}$ to C^1 problem one can substitute in Theorem 1, Necessary Conditions, the conclusion $\forall\, u \in \mathbb{R}^n : \forall\, (y^0,\, z^0) \in (f(x^0), g(x^0))'_u : \exists\, (\xi^0,\, \eta^0) \in C' \times K'[g(x^0)] \setminus \{(0,\, 0)\} : \langle \xi^0,\, y^0\rangle + \langle \eta^0,\, z^0\rangle \geq 0$ with $\exists\, (\xi^0,\, \eta^0) \in C' \times K'[g(x^0)] \setminus \{(0,\, 0)\} : \forall\, u \in \mathbb{R}^n : \langle \xi^0,\, f'(x^0)u\rangle + \langle \eta^0,\, g'(x^0)u\rangle \geq 0$.)
Take $x \in U \cap g^{-1}(-K)$, $x \neq x^0$, where U is the neighbourhood of x^0 determined by the local convexity properties of f and g, and put $u = x - x^0$. Like in the proof of Theorem 4 we get

$$\langle \xi^0,\, f(x) - f(x^0)\rangle + \langle \eta^0,\, g(x)\rangle \geq \langle \xi^0,\, f'(x^0)u\rangle + \langle \eta^0,\, g'(x^0)u\rangle \geq 0\,.$$

These inequalities show that x^0 is a minimizer of the function (8). According to Proposition 8 we have $\xi^0 \neq 0$. Now applying Theorem 4 we get that x^0 is a w-minimizer in sense I for (1). $\square$

We conclude with the following comments.

Theorem 4 obviously admits also other applications in addition to the demonstrated one, an observation which pleads for its significance. The given here version is not the unique and probably not the best reversal of Theorem 1, Necessary Conditions. It seems that the local Lipschitz assumption for f and g can be dropped. It seems also, that the convexity-type assumption for f can be dropped, if a slightly stronger convexity supposition for g is made.

The aim of Proposition 13 is rather to give an illustration of an application of Theorem 4, than to look for possibly weaker conditions guaranteeing that sense II w-minimizers are sense I w-minimizers. This can be observed, when comparing the obtained result with that of Proposition 5. Proposition 13 has in addition the assumption for f locally C-convex. In spite of this, we discover also some nuance in favor of Proposition 13, namely Proposition 5 deals with the stronger condition of g locally $\mathrm{int}\, K[g(x^0)]$-convex at x^0 versus the weaker condition g locally $K[g(x^0)]$-convex at x^0 in Proposition 13.

A more subtile application of a stronger version of Theorem 4 could bring a better result than this of Proposition 13. Still, let us express the opinion, that when used for comparison, the assumptions from the optimality conditions give limitations on the final result. For this reason, when comparing sense I and sense II concepts, it is better, if possible, to supply this matter with direct proofs, instead of treating it indirectly on the base of optimality conditions.

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