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Minty variational inequalities, increase-along-rays property and optimization[1]

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Abstract: Let $\mathbf{E}$ be a linear space, $K \subseteq \mathbf{E}$ and $f : K \to \mathbb{R}$. We put in terms of the lower Dini directional derivative a problem, referred to as $GMVI(f', K)$, which can be considered as a generalization of the Minty variational inequality of differential type (for short, $MVI(f', K)$). We investigate, in the case of K star-shaped (for short, st-sh), the existence of a solution x^* of $GMVI(f', K)$ and the property of f to increase-along-rays starting at x^* (for short, $f \in IAR(K, x^*)$). We prove that $GMVI(f', K)$ with radially l.s.c. function f has a solution $x^* \in \ker K$ if and only if $f \in IAR(K, x^*)$. Further, we prove, that the solution set of $GMVI(f', K)$ is a convex and radially closed subset of $\ker K$. We show also that, if $GMVI(f', K)$ has a solution $x^* \in K$, then x^* is a global minimizer of the problem $f(x) \to \min$, $x \in K$. Moreover, we observe that the set of the global minimizers of the related optimization problem, its kernel, and the solution set of the variational inequality can be different. Finally, we prove, that in case of a quasi-convex function f, these sets coincide.

Key words: Minty variational inequality, Generalized variational inequality, Existence of solutions, Increase along rays, Quasi-convex functions.

Math. Subject Classification: 90C30, 49J52, 49J40.

1 Introduction

Variational inequalities (for short, VI) provide a suitable mathematical model for a range of practical problems, and in particular equilibrium ones, see e. g. [1] or [2]. Generalizations towards vector VI were initiated in [3]; for recent results and survey on this field see [4] and [5]. When the operator involved in a VI has a primitive f, then we refer to the considered VI as of differential type. This kind of VI are widely studied because of their relation to optimization problems. Minty VI [6] (for short, MVI) of differential type, denoted $MVI(f', K)$, characterize a kind of equilibria more qualified than Stampacchia VI [7]. This leads to argue that when $MVI(f', K)$ admits a solution, then f has some regularity property. In this paper, we consider a generalization of MVI of differential type and show that, for such class of VI, the existence of a solution x^* is inherent only to functions f which increase-along-rays starting at x^*.

Throughout the paper, unless differently specified, we denote by $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ the extended real line, by $\mathbf{E}$ a real linear space and by K a nonempty subset of $\mathbf{E}$. Let $f : K \to \mathbb{R}$ be a given function. Then we denote by $\bar{f} : \mathbf{E} \to \overline{\mathbb{R}}$ the extension of f on $\mathbf{E}$ defined by $\bar{f}(x) = +\infty$ for $x \in \mathbf{E} \setminus K$. The lower Dini directional derivative (for short, Dini derivative) of f at $x \in K$ in the direction $d \in \mathbf{E}$ (as an element of $\overline{\mathbb{R}}$) is

$$f'_-(x, d) := \liminf_{t \to +0} \frac{1}{t} (\bar{f}(x + td) - f(x)) \tag{1}$$

(observe that, outside of the set K, the values of f are substituted by those of $\bar{f}$). Now we introduce the following problem

$$GMVI(f', K) \qquad f'_-(x,\, x^* - x) \le 0\,,\ \forall\, x \in K\,,$$

which consists in finding $x^* \in K$, such that $GMVI(f', K)$ holds. This problem somehow generalizes MVI of differential type

$$MVI(f', K) \qquad \langle f'(x),\, x^* - x \rangle \le 0\,,\ \forall\, x \in K\,,$$

where $K \subseteq \mathbb{R}^n$ and the function f is real-valued and Fréchet differentiable on an open set containing K. The problem is to find $x^* \in K$ for which $MVI(f', K)$ holds. Here $f'(x)$ stands for the Fréchet derivative of f and then $\langle f'(x), x^* - x \rangle$ is the directional derivative of f at x along the direction $x^* - x$. For this reason we use similar notation to quote the generalized problem $GMVI(f', K)$.

We investigate, in the case of K star-shaped (for short, st-sh), the existence of a solution x^* of $GMVI(f', K)$ and the property of f to increase along rays starting at x^* (for short, $f \in IAR(K, x^*)$). We prove, that $GMVI(f', K)$ with radially l.s.c. function f has a solution $x^* \in \ker K$ if and only if $f \in IAR(K, x^*)$. Here $\ker K$ denotes the kernel of the set K (see Rubinov [8]). We prove, that the solution set of $GMVI(f', K)$ is a convex and radially closed subset of $\ker K$. We show also that, if $GMVI(f', K)$ has a solution $x^* \in K$, then x^* is a global minimizer of the problem

$$\min f(x) \quad \text{s.t.} \quad x \in K\,, \tag{2}$$

and the existence of such a solution implies that the level sets of f are st-sh.

Since a function f is quasi-convex if and only if its level sets are convex, the functions with st-sh level sets can be considered as a generalization of the quasi-convex functions. Hence a natural question arises, whether or not when K is convex and $GMVI(f', K)$ has a solution, then f is quasi-convex. We give an example, which answers in the negative this question. Moreover, we observe that the set of the global minimizers of the related optimization problem, its kernel, and the solution set of the VI can be different. Finally, we prove that, in case of a quasi-convex function f, these sets coincide.

In the present paper, we relate the existence of solutions of $GMVI(f', K)$ to the properties: f with st-sh level sets, or f quasi-convex. These are properties of generalized convexity type, which are treated usually by means of the tools of nonsmooth analysis. Relations of VI to generalized convexity and nonsmooth analysis are often investigated, see e. g. Mastroeni [9] (also for vector VI), Mordukhovich [10], Thach, Kojima [11], Yang [12]. The existing results open perspectives for implementations in GMVI.

2 Existence of solutions and increase-along-rays property

In this section we investigate the relation between existence of solutions of $GMVI(f', K)$ and increase along-rays-property of f. We need the following definitions.

Definition 2.1 The set $\ker K := \{x \in K : [x^*, x] \subseteq K, \forall x \in K\}$ is called the *kernel* of K. The set K is called *star-shaped* iff $\ker K \neq \emptyset$.

Let x and x^* be two points in the space $\mathbf{E}$. We denote by $R_{x^*x} := \{x(t) = (1-t)x^* + tx, t \geq 0\}$ a ray starting at x^*.

Definition 2.2 Let $K \subseteq \mathbf{E}$ be a st-sh set, and $x^* \in \ker K$. The function $f : K \to \mathbb{R}$ (or $f : K \to \overline{\mathbb{R}}$) is said to *increase in K along rays starting at x^**, iff the restriction of f on the intervals $K_{x^*x} := K \cap R_{x^*x}$ is increasing, that is $0 \leq t_1 < t_2$ implies $f(x(t_1)) \leq f(x(t_2))$. The class of these functions is denoted by $IAR(K, x^*)$. If $K = \mathbf{E}$, then f is said to increase-along-rays starting at x^*.

Definition 2.3 Let $K \subseteq \mathbf{E}$ be a st-sh set, and $x^* \in \ker K$. The function $f : K \to \mathbb{R}$ is said to be *radially lower semi-continuous in K along rays starting at x^** (for short, $f \in RLSC(K, x^*)$), iff $\forall x \in K$ the restriction of f on the interval $K_{x^*x} := K \cap R_{x^*x}$ is lower semi-continuous (for short, l.s.c.). In other words, $f \in RLSC(K, x^*)$, iff $\forall x \in K$ the function $\varphi : [0, 1] \to \mathbb{R}$, $\varphi(t) := f((1-t)x^* + tx)$ is l.s.c.

We now give a different proof of Theorem 2 in [13], since its steps will be recalled in the proof of Theorem 3.1 below. This result links the existence of solutions of $GMVI(f', K)$ and the increase-along-rays property.

Theorem 2.1 Let $K \subseteq \mathbf{E}$ be a st-sh set, and $f : K \to \mathbb{R}$. If $x^* \in \ker K$ is a solution of $GMVI(f', K)$ and $f \in RLSC(K, x^*)$, then $f \in IAR(K, x^*)$. Conversely, if $x^* \in \ker K$ and $f \in IAR(K, x^*)$, then x^* is solution of $GMVI(f', K)$.

Proof Let $x^* \in \ker K$ be a solution of $GMVI(f', K)$ and $f \in RLSC(K, x^*)$. Fix $x \in K$ and denote $x(t) = (1-t)x^* + tx$, $0 \le t \le 1$. Observe first, that $f'_-(x(t),\, x^* - x) \le 0$ for $0 \le t \le 1$. This follows applying the positive homogeneity of the Dini derivative

$$f'_-(x(t),\, x^* - ((1-t)x^* + tx)) = t\, f'_-(x(t),\, x^* - x) \le 0\,.$$

We prove now that $f \in IAR(K, x^*)$. Let $0 \le t_1 < t_2$. Define the function $\varphi : [t_1,\, t_2] \to \mathbb{R}$ by

$$\varphi(t) = f((1-t)x^* + tx) - \frac{t_2 - t}{t_2 - t_1}\, f(x(t_1)) - \frac{t - t_1}{t_2 - t_1}\, f(x(t_2))\,.$$

Since φ is l.s.c., according to the Weierstrass Theorem, it attains its global minimum at some point $\hat{t} \in [t_1,\, t_2]$. We may assume that $\hat{t} \neq t_1$. Indeed, we have $\varphi(t_1) = \varphi(t_2) = 0$ and therefore, if the global minimum is achieved at $t = t_1$, it is attained also at $\hat{t} = t_2$. Denote now $\hat{x} = x(\hat{t})$. From the definition of $\hat{t}$ we have $\varphi'_-(\hat{t},\, -1) \ge 0$. Now

$$\begin{aligned}
\varphi'_-(\hat{t},\, -1) &= \liminf_{s \to +0} \frac{1}{s}\, \left(\varphi(\hat{t} - s) - \varphi(\hat{t})\right) \\
&= \liminf_{s \to +0} \frac{1}{s}\, \left(f(\hat{x} + s(x^* - x)) - f(\hat{x})\right) \\
&\quad - \frac{1}{t_2 - t_1}\, f(x(t_1)) + \frac{1}{t_2 - t_1}\, f(x(t_2)) \\
&= f'_-(\hat{x},\, x^* - x) - \frac{1}{t_2 - t_1}\, f(x(t_1)) + \frac{1}{t_2 - t_1}\, f(x(t_2)) \ge 0\,.
\end{aligned}$$

Therefore

$$f(x(t_2)) - f(x(t_1)) \ge -\,(t_2 - t_1)\, f'_-(\hat{x},\, x^* - x) \ge 0\,,$$

that is $f \in IAR(K, x^*)$.

To prove the converse, let $x^* \in \ker K$ and $f \in IAR(K, x^*)$. We must show that x^* is a solution of $GMVI(f', K)$. This is true, since for a fixed $x \in K$ we have

$$\begin{aligned} f'_-(x, x^* - x) &= \liminf_{t \to +0} \frac{1}{t} \left(f(x + t(x^* - x)) - f(x) \right) \\ &= \liminf_{t \to +0} \frac{1}{t} \left(f(x(1-t)) - f(x(1)) \right) \le 0 \, . \end{aligned}$$

□

Theorem 2.2 Let $K \subseteq \mathbf{E}$ be a st-sh set and $f : K \to \mathbb{R}$. If $x^* \in \ker K$ is a solution of $GMVI(f', K)$ and $f \in RLSC(K, x^*)$, then $x^* \in GM(f, K)$, which is the set of global minimizers of problem (2).

Proof Let $x^* \in \ker K$ be a solution of $GMVI(f', K)$ and $x \in K$. According to Theorem 2.1, f increases along $R_{x^* x} \cap K$, whence $f(x^*) \le f(x)$. Therefore $x^* \in GM(f, K)$. □

Next result establishes a relation between existence of solutions of $GMVI(f', K)$ and star-shapedness of the level sets of f, defined by $\mathrm{lev}_c f = \{x \in K : \, f(x) \le c\}$, for $c \in \mathbb{R}$.

Theorem 2.3 Let $K \subseteq \mathbf{E}$ be a st-sh set and $f : K \to \mathbb{R}$. If there exists a solution $x^* \in \ker K$ of $GMVI(f', K)$ and $f \in RLSC(K, x^*)$, then all the level sets of f are st-sh. In particular the set $GM(f, K)$ of the global minimizers of problem (2) is st-sh.

Proof According Theorem 2.2, $x^* \in GM(f, K)$. Fix $c \in \mathbb{R}$ and consider the level set $\mathrm{lev}_c f$. If $c < f(x^*)$, then $\mathrm{lev}_c f$ is empty, hence st-sh. Let now $c \ge f(x^*)$. Fix $x \in \mathrm{lev}_c f$ and let $x(t) = (1-t)x^* + tx$. We must show that $f(x(t)) \le c$. According to Theorem 2.1, $f \in IAR(K, x^*)$. This implies $f(x(t)) \le f(x(1)) = f(x) \le c$. □

In Theorem 2.1 the assumption $f \in RLSC(K, x^*)$ appears in only one of the two opposite implications. A natural question arises, whether or not it can be dropped at all. The next example answers this question in the negative.

Example 2.1 Define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 0$ for $x = 0$ or x irrational, and $f(x) = -q$ for $x \neq 0$ rational with $x = p/q$, $q > 0$ and p and q relatively prime. The function f is not l.s.c. . The Dini derivatives are $f'(x, u) = -\infty$ for each $x \in \mathbb{R}$ and $u \in \mathbb{R} \setminus \{0\}$. Consequently, each point $x^* \in \mathbb{R}$ is a solution of $GMVI(f', K)$. At the same time f has no global minimizers. In particular $x^0 = 0$ is among the solutions of $GMVI(f', K)$, which is a global maximizer of f. Even more: while having no global minimizers, there is a set of points which is dense in $\mathbb{R}$, namely the set of the irrational numbers, each of which is both a solution of $GMVI(f', K)$ and a global maximizer of f.

3 Existence of solutions and star-shapedness

Example 2.1 and Theorem 2.1 point out the importance of the $RLSC$ property of f (i.e. the property of f to be radially l.s.c.) for the existence of solutions of $GMVI(f', K)$. Theorem 2.1 discusses however only $GMVI(f', K)$ with a st-sh set K and solutions $x^* \in \ker K$. In this section we show, that these requirements are natural, in the sense that, under the $RLSC$ property of f, the existence of solutions is habitual only to $GMVI(f', K)$ with K st-sh. Moreover, there do not exist solutions x^* outside $\ker K$. In order to come to this conclusion, we extend first problem $GMVI(f', K)$.

In the Introduction we defined the extended function $\bar{f} : \mathbf{E} \to \mathbb{R} \cup \{+\infty\}$ in a way that $\bar{f}$ coincides with f on K and is $+\infty$ on $\mathbf{E} \setminus K$. Together with $GMVI(f', K)$ we consider the following problem

$$GMVI(\bar{f}', \mathbf{E}) \qquad\qquad \bar{f}'_-(x,\, x^* - x) \leq 0, \;\; x \in \mathbf{E}\,.$$

The meaning of problem $GMVI(\bar{f}', \mathbf{E})$ is the following. The task is to find $x^* \in \mathbf{E}$ such that the inequality $\bar{f}'_-(x,\, x^* - x) \leq 0$ is satisfied any time the directional derivative $\bar{f}'_-(x,\, x^* - x)$ has sense. For the latter we make the following comment. The derivative $\bar{f}'_-(x,\, u)$ is given by the right hand side of (1). In $\mathbb{R} \cup \{\pm\infty\}$ the operation $+\infty - (+\infty)$ is not defined. Consequently, we will accept that $\bar{f}'_-(x,\, u)$ has no sense if and only if $f(x) = +\infty$ and there exists an interval $(x,\, x + \varepsilon\, u) = \{x + tu :\ 0 < t < \varepsilon\}$ with $\varepsilon > 0$ contained in $\mathbf{E} \setminus \operatorname{dom} \bar{f}$ (recall that $\operatorname{dom} \bar{f} = \{x \in \mathbf{E} :\ \bar{f}(x) \in \mathbb{R}\}$).

We extended problem $GMVI(f', K)$ to $GMVI(\bar{f}', \mathbf{E})$. Simultaneously, we can extend problem (2) to the optimization problem

$$\min \bar{f}(x)\,, \quad \text{s.t.} \quad x \in \mathbf{E}\,. \tag{3}$$

Next proposition gives the relation between these pairs of problems.

Proposition 3.1 Let $\bar{f} : \mathbf{E} \to \mathbb{R} \cup \{+\infty\}$ be a given function and set $K := \operatorname{dom} \bar{f}$ and $f := \bar{f}|K$ (the restriction of $\bar{f}$ on K). Then $GMVI(\bar{f}', \mathbf{E})$ has a solution $x^* \in K$ if and only if x^* solves $GMVI(f', K)$. Similarly, problem (3) has a global (local) minimizer $x^* \in K$ if and only x^* is a global (local) minimizer for problem (2).

Proof Let $x^* \in K := \operatorname{dom} \bar{f}$ be a solution of $GMVI(\bar{f}', \mathbf{E})$. Then x^* is a solution of $GMVI(f', K)$, since all the inequalities in $GMVI(f', K)$ enter also $GMVI(\bar{f}', \mathbf{E})$. Let now $x^* \in K$ be a solution of $GMVI(f', K)$. In order to state that x^* solves $GMVI(\bar{f}', \mathbf{E})$, we must check that for $x \in \mathbf{E} \setminus K$ the inequality $\bar{f}_-(x,\, x^* - x) \le 0$ is satisfied, assuming the Dini derivative has sense. But this is checked rather trivially. If the Dini derivative has sense, then there exists a sequence $t_k \to +0$, such that $\bar{f}(x + t_k(x^* - x))$ is finite. Then

$$\bar{f}'_-(x,\, x^* - x) \le \liminf_k \frac{1}{t_k} \left(\bar{f}(x + t_k(x^* - x)) - f(x)\right) = -\infty < 0\,.$$

The part concerning the equivalency of the optimization problems (2) and (3) is obvious. □

Definition 3.1 Let $\bar{f} : \mathbf{E} \to \mathbb{R} \cup \{+\infty\}$ be a given function. Then we say that $\bar{f}$ has the $RLSC$ property, denoted $\bar{f} \in RLSC$, iff $\bar{f} \in RLSC(\mathbf{E}, x^*)\ \forall x^* \in \mathbf{E}$.

Definition 3.2 Let $K \subseteq \mathbf{E}$ and $x^* \in \mathbf{E}$. We say that K is *radially closed along the rays starting at* x^* iff $K \cap R_{x^*x}$ is closed in R_{x^*x} (in view of this parameterization, the topological structure on the rays is determined by the topological structure on $\mathbb{R}$). We say, that K is *radially closed* iff the previous property holds for every $x^* \in \mathbf{E}$.

Proposition 3.2 Let $\bar{f} : \mathbf{E} \to \mathbb{R} \cup \{+\infty\}$ be a given function and $K := \operatorname{dom} \bar{f}$ be radially closed. Then $\bar{f} \in RLSC$ if and only if $f \in RLSC(K, x^*)$, $\forall x^* \in K$.

Proof The assumption $\bar{f} \in RLSC$ implies that the restriction of $\bar{f}$ on R_{x^*x} is radially l.s.c. (x^* and x arbitrary) and moreover, it is radially l.s.c. on $R_{x^*x} \cap K$, no matter whether K is radially closed or not.

Conversely, let $f \in RLSC(K, x^*)$. We must show that $\bar{f} \in RLSC(\mathbf{E}, x^*)$. Let $x^k = (1 - t_k)x^* + t_k x$, $t_k \to t_0$ and $x^0 = (1 - t_0)x^* + t_0 x$. We must check that $\bar{f}(x^0) \leq \liminf_k \bar{f}(x^k)$. The points x^k at which $\bar{f}(x^k) = +\infty$, obviously do not affect this inequality. Therefore, the only case which has to be checked, is when $x^k \in K$. Since K is radially closed, it follows that also $x^0 \in K$. From $f \in RLSC(K, x^*)$, it follows that $\bar{f}(x^0) = f(x^0) \leq \liminf_k f(x^k) = \liminf_k \bar{f}(x^k)$. $\square$

The following theorem extends Theorem 2.1 to $GMVI(\bar{f}', \mathbf{E})$.

Theorem 3.1 Let $\bar{f} : \mathbf{E} \to \mathbb{R} \cup \{+\infty\}$ be a given function, and $K := \operatorname{dom} \bar{f}$ be radially closed. If $x^* \in \operatorname{dom} \bar{f}$ is a solution of $GMVI(\bar{f}', \mathbf{E})$ and $\bar{f}$ has the $RLSC$ property, then $\bar{f} \in IAR(\mathbf{E}, x^*)$. Conversely, if $x^* \in \operatorname{dom} \bar{f}$ and $\bar{f} \in IAR(\mathbf{E}, x^*)$, then x^* is a solution of $GMVI(\bar{f}', \mathbf{E})$.

Proof Let $x^* \in K$ be a solution of $GMVI(\bar{f}', \mathbf{E})$ and $\bar{f} \in RLSC$. Fix $x \in \mathbf{E}$ and let $x(t) = (1-t)x^* + tx$, $0 \leq t \leq 1$. Observe that $\bar{f}_-(x(t),\, x^* - x) \leq 0$ for $0 \leq t \leq 1$, as far as this derivative has sense, which follows in the same way as in the proof of Theorem 2.1. Next we prove, that $\bar{f} \in IAR(\mathbf{E}, x^*)$. This follows by constructing a function φ like in the proof of Theorem 2.1 and concluding in the same way as there, that $0 \leq t_1 < t_2$ and $x(t) \in K$ for $t_1 \leq t \leq t_2$, implies

$f(x(t_2)) - f(x(t_1)) \geq 0$. We claim, and this is the main difference from the proof of Theorem 2.1, that if $x(t)$ has left K, it will not return back. Assume that this is not the case. Then, since K is radially closed (we recall, that the complement of a closed set on the real line, is a union of open intervals), it follows that there exists $\delta > 0$ and $t_0 > 0$, such that $0 < t_0 - \delta < t_0$, $\bar{f}(x(t)) = +\infty$ for $t_0 - \delta < t < t_0$, and $\bar{f}(x(t_0)) < +\infty$. Then

$$f'_-(x(t_0), x^* - x(t_0)) = t_0\, f'_-(x(t_0), x^* - x) = t_0 \liminf_{s \to +0} \frac{1}{s} \left(f(x(t_0 - s)) - f(x(t_0)) \right) = +\infty \,,$$

which contradicts to x^* solution of $GMVI(\bar{f}', \mathbf{E})$.

Conversely, let $x^* \in K$ and $\bar{f} \in IAR(\mathbf{E}, x^*)$. We must show that x^* is a solution of $GMVI(\bar{f}', \mathbf{E})$. For a fixed $x \in K$ this is true as in the proof of Theorem 2.1. Let now $x \in \mathbf{E} \setminus K$. Since K is radially closed, it follows that there exists $\delta \in (0, 1)$, such that $x(t) = (1-t)x^* + tx \in \mathbf{E} \setminus K$ for $1 - \delta < t \leq 1$. Therefore the derivative

$$f'_-(x,\, x^* - x) = \liminf_{s \to +0} \frac{1}{s} \left(f(x(1) + s(x^* - x)) - f(x(1)) \right) = \liminf_{s \to +0} \frac{1}{s} \left(f(x(1-s)) - f(x(1)) \right)$$

has no sense. $\square$

Proposition 3.3 Let $\bar{f} : \mathbf{E} \to \mathbb{R} \cup \{+\infty\}$ be a given function and assume $\bar{f} \in IAR(\mathbf{E}, x^*)$. Then, x^* is a global minimizer of problem (3), the function $\bar{f}$ has st-sh level sets and each such nonempty set contains x^* in its kernel. Furthermore, the set $K := \operatorname{dom} \bar{f}$ is st-sh too and $x^* \in \ker K$.

Proof Fix $c \in \overline{\mathbb{R}}$ and consider the level set $\mathrm{lev}_c \bar{f}$. If $c < \bar{f}(x^*)$, then $\mathrm{lev}_c \bar{f}$ is empty, hence st-sh. Let now $c \geq f(x^*)$. Fix $x \in \mathrm{lev}_c \bar{f}$ and let $x(t) = (1-t)x^* + tx$. From $\bar{f} \in IAR(\mathbf{E}, x^*)$, for $0 \leq t \leq 1$ we have $\bar{f}(x^*) = \bar{f}(x(0)) \leq \bar{f}(x(t)) \leq \bar{f}(x(1)) = \bar{f}(x) \leq c$, whence x^* is a global minimizer of problem (3) and $[x^*,\, x] \subseteq \mathrm{lev}_c \bar{f}$. The set K is st-sh and $x^* \in \ker K$, since $K = \bigcup \{\mathrm{lev}_c \bar{f} \,:\, c \in \mathbb{R}\}$. $\square$

Theorem 3.2 Let $\bar{f} : \mathbf{E} \to \mathbb{R} \cup \{+\infty\}$ be a given function enjoying the $RLSC$ property and $K := \operatorname{dom} \bar{f}$ be radially closed. Assume that $GMVI(\bar{f}', \mathbf{E})$ has at least one solution $x^* \in K$. Then, all the level sets of $\bar{f}$ are st-sh and contain x^* in their kernels. In particular, the set $GM(\bar{f}, \mathbf{E})$ of the global minimizers of problem (3) is st-sh too and $x^* \in \ker GM(\bar{f}, \mathbf{E})$. Further, the set K is st-sh and $x^* \in \ker K$. Moreover, the set SOL of the solutions of $GMVI(\bar{f}', \mathbf{E})$ is a convex subset of $\ker GM(\bar{f}, \mathbf{E})$ and SOL is radially closed.

Proof Let $x^* \in K$ be a solution of $GMVI(\bar{f}', \mathbf{E})$. According to Theorem 3.1, $\bar{f} \in IAR(\mathbf{E}, x^*)$. Now the conclusion follows from Proposition 3.3. In particular $GM(\bar{f}, \mathbf{E})$ is st-sh and $x^* \in \ker GM(\bar{f}, \mathbf{E})$, since $GM(\bar{f}, \mathbf{E}) = \operatorname{lev}_{\bar{f}(x^*)} \bar{f}$.

In fact, we have shown that $SOL \subseteq \ker GM(\bar{f}, \mathbf{E})$. To prove that the solution set SOL is convex, take $x_0^*,\, x_1^* \in SOL$ and let $x^* = (1 - t_*)x_0^* + t_* x_1^*$, where $t_* \in (0,\, 1)$ is a fixed number.

We have $x^* \in K$. Indeed, $x_0^* \in SOL$ and from Theorem 3.1 we have $\bar{f} \in IAR(\mathbf{E}, x_0^*)$. Therefore $\bar{f}(x^*) \le \bar{f}(x_1^*) < +\infty$ (the inequality $f(x_1^*) < +\infty$ follows from $x_1^* \in SOL$ and the definition of the solution set).

Now we show that $\bar{f} \in IAR(\mathbf{E}, x^*)$. Let $x \in \mathbf{E}$ and $x(t) = (1 - t)x^* + tx$. We must show that $0 \le t_1 < t_2$ implies $\bar{f}(x(t_1) \le \bar{f}(x(t_2))$. For this purpose we define, for $t \ge 0$, the functions

$$x_0(t) = (1 - t)x_0^* + tx(t_1)\,, \quad x_1(t) = (1 - t)x_1^* + tx(t_2)\,.$$

Taking

$$\tau = \frac{t_2}{t_1 + (t_2 - t_1)t_*}\,, \quad \sigma = \frac{t_1}{t_1 + (t_2 - t_1)t_*}\,,$$

we check easily that $x_0(\tau) = x_1(\sigma)$, both sides of this equality are equal to

$$\frac{t_1(1 - t_2)(1 - t_*)}{t_1 + (t_2 - t_1)t_*}\, x_0^* + \frac{(1 - t_1)t_2 t_*}{t_1 + (t_2 - t_1)t_*}\, x_1^* + \frac{t_1 t_2}{t_1 + (t_2 - t_1)t_*}\, x\,,$$

and $\tau > 1 > \sigma \ge 0$. The latter comes from

$$\tau - 1 = \frac{(t_2 - t_1)(1 - t_*)}{t_1 + (t_2 - t_1)t_*} > 0\,, \quad 1 - \sigma = \frac{(t_2 - t_1)t_*}{t_1 + (t_2 - t_1)t_*} > 0\,.$$

Now, since $\bar{f} \in IAR(\mathbf{E}, x_0^*)$ and $\bar{f} \in IAR(\mathbf{E}, x_1^*)$, which follows from Theorem 3.1 with the account of x_0^*, $x_1^* \in SOL$, we get

$$\bar{f}(x(t_1)) = \bar{f}(x_0(1)) \leq \bar{f}(x_0(\tau)) = \bar{f}(x_1(\sigma)) \leq \bar{f}(x_1(1)) = \bar{f}(x(t_2))\,.$$

Thus, $\bar{f} \in IAR(\mathbf{E}, x^*)$. Applying now Proposition 3.3, we get $x^* \in SOL$. Therefore, the solution set SOL is convex.

In order to show that SOL is radially closed, fix x^*, $\tilde{x} \in \mathbf{E}$ and let $\tilde{x}(t) = (1-t)x^* + t\tilde{x}$. We have to prove that, if $t_k \to t_0$, $t_k \geq 0$, and $\tilde{x}(t_k) \in SOL$, then also $\tilde{x}(t_0) \in SOL$. In view of Theorem 3.1, for this purpose, it is enough to show that $\bar{f} \in IAR(\mathbf{E}, \tilde{x}(t_0))$. Fix $x \in \mathbf{E}$ and denote $x(t) = (1-t)\tilde{x}(t_0) + tx$. Let $0 \leq t' < t''$. We must show that $\bar{f}(x(t')) \leq \bar{f}(x(t''))$. For this purpose we define $x^k(t) = (1-t)\tilde{x}(t_k) + tx(t'')$ and $x^0(t) = (1-t)\tilde{x}(t_0) + tx(t'')$. From $\tilde{x}(t_k) \in SOL$, it follows, that $\bar{f} \in IAR(\mathbf{E}, \tilde{x}(t_k))$, hence

$$\bar{f}(x(t'')) = \bar{f}(x^k(1)) \geq \bar{f}(x^k(\frac{t'}{t''}))$$

and so $\bar{f}(x(t'')) \geq \liminf_{k\to+\infty} \bar{f}(x^k(\frac{t'}{t''})) \geq \bar{f}(x^0(\frac{t'}{t''})) = \bar{f}(x(t'))$. In the inequality $\liminf_{k\to+\infty} \bar{f}(x^k(\frac{t'}{t''})) \geq \bar{f}(x^0(\frac{t'}{t''}))$ we have used that all the points $x^k(\frac{t'}{t''})$ lay on ray with initial point $(1-t')x^* + t'x$ and passing through $(1-t')\tilde{x} + t'x$, that $x^k(\frac{t'}{t''}) \to x^0(\frac{t'}{t''})$, and $\bar{f} \in RLSC$. Thus, $\bar{f} \in IAR(\mathbf{E}, \tilde{x}(t_0))$, and according to Proposition 3.3, we have $\tilde{x}(t_0) \in SOL$.

□

We conclude this section considering the case when $\mathbf{E}$ is a topological vector space (for short, t.v.s.), $\bar{f} : \mathbf{E} \to \mathbb{R} \cup \{+\infty\}$ is lower semi-continuous, and $K := \operatorname{dom} \bar{f}$ is a closed set. Next theorem is a reformulation of Theorem 3.2 for this case.

Theorem 3.3 Let $\mathbf{E}$ be a t.v.s., $\bar{f} : \mathbf{E} \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function and $K := \operatorname{dom} \bar{f}$ be closed. Assume that $GMVI(\bar{f}', \mathbf{E})$ has at least one solution $x^* \in K$. Then all the level sets of $\bar{f}$ are st-sh and contain x^* in their kernels. In particular, the set $GM(\bar{f}, \mathbf{E})$ of the global minimizers of problem (3) is st-sh too and $x^* \in \ker GM(\bar{f}, \mathbf{E})$. Further, the set K is st-sh and $x^* \in \ker K$. Moreover, the set SOL of the solutions $x^* \in K$ of $GMVI(\bar{f}', \mathbf{E})$ is a convex subset of $\ker GM(\bar{f}, \mathbf{E})$ and SOL is a closed set.

Proof The function $\bar{f}$ has the RLSC property, since the restriction of a l.s.c. function on a closed set, in particular on a ray, is a l.s.c. function. The set K is radially closed, since a finite intersection of closed sets, in particular of K and a ray, is a closed set. Therefore, the hypotheses of Theorem 3.2 are satisfied, hence also its conclusions hold. They coincide with the conclusions of the present theorem, except in one case. Namely, the conclusion SOL is a closed set needs a separate proof.

In order to show that SOL is closed, choose a sequence $x^k \to x^0$, $x^k \in SOL$. We have to prove that $x^0 \in SOL$. In view of Theorem 3.1, for this purpose it is enough to show that $\bar{f} \in IAR(\mathbf{E}, \tilde{x}(t_0))$. Fix $x \in \mathbf{E}$ and let $x(t) = (1-t)\tilde{x}(t_0) + tx$. For $0 \le t' < t''$, we must show that $\bar{f}(x(t')) \le \bar{f}(x(t''))$. For this purpose we define $x^k(t) = (1-t)x^k + tx(t'')$ and $x^0(t) = (1-t)x^0 + tx(t'')$. From $x^k \in SOL$, it follows that $\bar{f} \in IAR(\mathbf{E}, x^k)$, whence

$$\bar{f}(x(t'')) = \bar{f}(x^k(1)) \ge \bar{f}(x^k(\frac{t'}{t''}))$$

and so $\bar{f}(x(t'')) \ge \liminf_{k\to+\infty} \bar{f}(x^0(\frac{t'}{t''})) \ge \bar{f}(x^0(\frac{t'}{t''})) = \bar{f}(x(t'))$. In the inequality $\liminf_{k\to+\infty} \bar{f}(x^k(\frac{t'}{t''})) \ge \bar{f}(x^0(\frac{t'}{t''}))$ we have used that $x^k(\frac{t'}{t''}) \to x^0(\frac{t'}{t''})$ and that $\bar{f}$ is l.s.c. Thus, $\bar{f} \in IAR(\mathbf{E}, x^0)$ and according to Proposition 3.3, we have $\bar{x}^0 \in SOL$. □

4 Comparison of MVI, GMVI and the related optimization problem

In the notation of Theorem 3.3 we see that the following inclusions take place: $SOL \subseteq \ker GM(\bar{f}, \mathbf{E}) \subseteq GM(\bar{f}, \mathbf{E})$. Next example shows that these inclusions can be strict.

Example 4.1 The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x_1, x_2) = \begin{cases} \dfrac{\min(|x_1| - 1, |x_2| - 1)}{\max(|x_1| - 1, |x_2| - 1)} & , \quad \min(|x_1|, |x_2|) > 1\,, \\[2ex] 0 & , \quad \min(|x_1|, |x_2|) \le 1\,, \end{cases}$$

is l.s.c. The solution set SOL of $GMVI(f', \mathbb{R}^2) = GMVI(\bar{f}', K)$, with $K = \mathbb{R}^2$ and $\bar{f} = f$, is the singleton $SOL = \{(0,\,0)\}$. The square $\ker GM(f, \mathbb{R}^2) = \{(x_1, x_2) \,:\, \max(|x_1|, |x_2|) \le 1\}$ is the kernel of the set $GM(f, \mathbb{R}^2) = \{(x_1, x_2) \,:\, \min(|x_1|, |x_2|) \le 1\}$ of the global minimizers of (3). All these three sets are different.

The function f is l.s.c., since its level sets are closed. The structure of the set $GM(f, \mathbb{R}^2)$ and its kernel follow immediately from the definition of f. To determine the set SOL we use $x^* \in SOL$ iff $f \in IAR(\mathbb{R}^2, x^*)$, or equivalently, if the function $t \to \varphi(t) = f((1-t)x_1^* + tx_1, (1-t)x_2^* + tx_2)$, $t \ge 0$, is increasing for each $x = (x_1, x_2) \in \mathbb{R}^2$. For $x^* = (0,\,0)$ and $|x_1| \le |x_2|$ (from the symmetry, the case $|x_1| > |x_2|$ is treated similarly) this function is

$$\varphi(t) = \begin{cases} \dfrac{t\,|x_1| - 1}{t\,|x_2| - 1} & , \qquad t > \dfrac{1}{|x_1|}\,, \\[2ex] 0 & , \quad 0 \le t \le \dfrac{1}{|x_1|}\,, \end{cases}$$

and for $t > 1/\min(|x_1|, |x_2|)$ has derivative $\varphi'(t) = (|x_2| - |x_1|)\,/\,(t|x_2| - 1)^2 \ge 0$. Hence, φ is increasing. For $x^* = (x_1^*,\, x_2^*) \ne (0,\,0)$, when $x_1^* \ne x_2^*$ we take $x = (2,\,2)$ and when $x_1^* = x_2^*$ we take $x = (2,\,-2)$. An easy calculation shows that then φ attains a strict maximum at $t = 1$, whence $f \notin IAR(\mathbb{R}^2, x^*)$. $\square$

Assume that f is differentiable on an open set containing K. A natural question arises: whether or not problem $MVI(f', K)$ is equivalent to $GMVI(f', K)$. The latter, according to Proposition 3.1, is equivalent to $GMVI(\bar{f}', \mathbf{E})$. The equivalence is understood in the sense of coincidence of the solution sets. As far as K is st-sh and only solutions $x^* \in \ker K$ are considered, the two problems $MVI(f', K)$ and $GMVI(f', K)$ remain equivalent. Indeed, one may observe that in the Dini directional derivative $f'_-(x, x^* - x)$, the direction $x^* - x$ does not lead outside the set K. In the case in which K is convex (which holds in particular, if f is quasi-convex on K), the two problems are equivalent. The situation changes when the points outside $\ker K$ are considered. According to Theorem 3.1 and Proposition 3.3, $GMVI(\bar{f}', \mathbf{E})$ cannot have solutions outside $\ker K$. This makes the solution sets of $GMVI(\bar{f}', \mathbf{E})$ and $MVI(f', K)$ not equivalent when considering solutions outside $\ker K$. The reason is that for $MVI(f', K)$, the directional derivative along directions leading outside the set K involves only the values of f, which is supposed to exist and to be differentiable on an open set containing K. On the contrary, problem $GMVI(\bar{f}', \mathbf{E})$ substitutes, for $x \notin K$, the values $f(x)$ by $\bar{f}(x) = +\infty$, which interferes the Dini directional derivatives in the directions leading outside the set K, and hence the solution set. This is illustrated in the following example.

Example 4.2 Consider $MVI(f', K)$ with $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x_1, x_2) = x_2^2$, and $K = \{(x_1, x_2) : x_1 \geq 1 \text{ or } (x_1 \geq -1 \text{ and } |x_2| \leq 1)\}$. Then it is easy to check that the solution set of $MVI(f', K)$ is $\{x \in \mathbb{R}^2 : x_1 \geq -1,\, x_2 = 0\}$ and the solution set of the corresponding $GMVI(\bar{f}', \mathbb{R}^2)$ (with $\bar{f}(x) = +\infty$ for $x \notin K$) is $\{x \in \mathbb{R}^2 : x_1 \geq 1,\, x_2 = 0\}$.

Theorem 3.2 states that any solution of $GMVI(f', K)$ is a solution of the related minimization problem. Now we show that this is not the case when we consider $MVI(f', K)$.

Example 4.3 Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as $f(x_1, x_2) = x_1 x_2 (x_1 + x_2) - \frac{1}{3}(x_1 + x_2 - 1)^2$ and let $K = \{[(-1, 0), (0, 1)] \cup [(0, 1), (0, 4)]\}$. The set K is st-sh with $\ker \mathrm{K} = \{(0, 1)\}$ and $MVI(f', K)$ has the unique solution $x^* = (-1, 0) \notin \ker \mathrm{K}$. This point does not belong to the set $GM(f, K)$ which is the singleton $\{(0, 4)\}$.

Remark 4.1 If $MVI(f', K)$ admits at least one solution $x^* \in \ker \mathrm{K}$, then each solution of $MVI(f', K)$ is a solution of the related minimization problem. In fact, assume that $\bar{x} \neq x^*$ is a solution of $MVI(f', K)$. Since $x^* \in \ker \mathrm{K}$ solves $MVI(f', K)$ then x^* solves $GMVI(f', K)$

and so $f \in IAR(K, x^*)$. Hence f is increasing along the ray $R_{x^*x} \cap K$. Since $\bar{x}$ also solves $MVI(f', K)$, then it is easily seen that f is increasing along the ray $\mathbf{E}_{\bar{x}x^*} \cap K$. This implies that all the points on the segment $[x^*, x]$ are minimizers of f over K.

5 The case of a quasi-convex function

If $GMVI(\bar{f}', \mathbf{E})$ has a solution and the hypotheses of Theorem 3.3 are satisfied, the conclusion is that $\bar{f}$ has st-sh level sets. Functions with st-sh level sets can be considered as some generalization of quasi-convex functions. Therefore, we wonder whether or not if $GMVI(\bar{f}', \mathbf{E})$ has a solution x^*, then $\bar{f}$ is quasi-convex. For a function of one variable, this result is a consequence of the increase along property. However, already for functions of two variables, Example 4.1 above answers in negative this question.

Still, we can investigate more carefully $GMVI(\bar{f}', \mathbf{E})$ with quasi-convex functions f for the following reason. The main justification to introduce a VI could be to have an alternative approach to the underlying optimization problem. The best opportunity is if the solution sets of the two problems coincide. In our case these are $GMVI(\bar{f}', \mathbf{E})$ and problem (3) with solution sets respectively SOL and $GM(\bar{f}, \mathbf{E})$. In Theorem 3.3, under the made there assumptions, we see that $SOL \subseteq GM(\bar{f}, \mathbf{E})$. Example 4.1 shows that, in general, these two sets are different. In the next theorem we prove, that when f is quasi-convex, the two sets coincide, which makes the quasi-convexity a rather natural assumption to treat optimization problems through VI of Minty type.

Theorem 5.1 Let $\mathbf{E}$ be a real linear space, $\bar{f} : \mathbf{E} \to \mathbb{R} \cup \{+\infty\}$ be a quasi-convex function having the RLSC property and $K := \operatorname{dom} \bar{f}$ be radially closed and convex. Then the solution set SOL of $GMVI(\bar{f}', \mathbf{E})$ and the set $GM(\bar{f}, \mathbf{E})$ of the global minimizers of problem (3) coincide.

Proof Applying Theorem 3.2, it remains to show that each $x^* \in GM(\bar{f}, \mathbf{E})$ is a solution of

$GMVI(\bar{f}', \mathbf{E})$. For this purpose, according to Theorem 3.1, it is sufficient to prove that $\bar{f} \in IAR(\mathbf{E}, x^*)$. Assume, on the contrary, that there exists $x \in \mathbf{E}$, such that $\bar{f}(x(t_1)) > \bar{f}(x(t_2))$ for some $0 \le t_1 < t_2$ (here $x(t) = (1-t)x^* + tx$). Put $c = \bar{f}(x(t_2))$. From $x^* \in GM(\bar{f}, \mathbf{E})$ it follows that $\bar{f}(x^*) \le \bar{f}(x(t_2)) = c$. Hence both $\bar{f}(x^*)$, $f(x(t_2)) \in \mathrm{lev}_c \bar{f}$, while $(1 - \frac{t_1}{t_2})x^* + \frac{t_1}{t_2} x(t_2) = x(t_1) \notin \mathrm{lev}_c \bar{f}$. This contradicts the quasi-convexity assumption. $\square$

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