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Well-posedness and scalarization in vector optimization*

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Abstract

In this paper we study several existing notions of well-posedness for vector optimization problems. We distinguish them into two classes and we establish the hierarchical structure of their relationships. Moreover, we relate vector well-posedness and well-posedness of an appropriate scalarization. This approach allows us to show that, under some compactness assumption, quasiconvex problems are well-posed.

Keywords: well-posedness, vector optimization problems, nonlinear scalarization, generalized convexity.

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1 Introduction

The notion of well-posedness plays a central role in stability theory for scalar optimization (see, e.g., [12]). In the last decades, some extensions of this concept to vector optimization appeared (see [13] and the references therein). Here we focus on some generalizations to the vector case of well-posedness in the sense of Tykhonov (see e.g. [3], [23], [5], [19], [11], [20], [21], [18] and [26]).

The aim of this paper is twofold. First, we list and classify some existing notions of well-posedness for vector optimization problems and we compare them. Further, we study the relationships between the well-posedness of a vector optimization problem and the well-posedness of an appropriate scalarized problem.

The classification proposed here essentialy groups the definitions into two main classes: pointwise and global well-posedness. The definitions of the first group consider a fixed efficient point (or the image of an efficient point) and deal with well-posedness of the vector optimization problem at this point. This approach imposes that the minimizing sequences related to the considered point are well-behaved. Since in the vector case the solution set is typically not a singleton, there is also a class of definitions that involve the efficient frontier as a whole. In Section 3, we compare the various definitions analyzing the hierarchical structure of their relationships.

The second part of the paper is devoted to the study of some properties of well-posed vector optimization problems. In Section 4, we underline that most of the vector well-posedness notions implicitly impose some requirements concerning the features of the solutions in the image space. This fact appears only in the framework of vector optimization, since in the scalar case well-posedness concerns only the interplay, through the objective function, between the behaviour of the minimizing sequences in the domain and the image space.

When we consider a scalarization of a vector optimization problem, a natural question arises: is the well-posedness of the vector problem equivalent to the well-posedness of the scalarized problem? In Section 5, we deal with a special scalarizing function (the so called "oriented distance"), already used in [27] (see also the references therein). This function allows us to establish a parallelism between the well-posedness of the original vector problem and the well-posedness of the associate scalar problem. Indeed, we show that one of the weakest notions of well-posedness in vector optimization is linked to the well-setness of the scalarized problem, while some stronger notion in the vector case is related to Tykhonov well-posedness of the associated scalarization.

These results constitute a simple tool to show that, under some additional compactness assumptions, quasiconvex vector optimization problems are well-posed. Thus, we can extend to vector optimization a known result about scalar problems [9] and improve a previous result concerning convex vector problems [26].

2 Problem setting and notation

Let X and Y be normed vector spaces. We denote by B the unit ball both in X and Y, since it will be clear to which space we refer. The space Y is endowed with an order relation given by a closed, convex and pointed cone K with nonempty interior, in the following way

$$y_1 \leq_K y_2 \Longleftrightarrow y_2 - y_1 \in K,$$
$$y_1 <_K y_2 \Longleftrightarrow y_2 - y_1 \in \ \mathrm{int}K.$$

We recall that a convex set $A \subseteq K$ is a *base* for the cone K when $0 \notin A$ and for every $k \in K$, $k \neq 0$, there are unique elements $a \in A,\ t > 0$, such that $k = ta$.

Let $f : X \to Y$ be a function and let $S \subseteq X$ be a subset of X. We consider the vector optimization problem (S, f) given by

$$\min f(x)\,, \quad x \in S.$$

Throughout this paper we will assume that the objective function f in problem (S, f) is continuous. A point $x_0 \in S$ is an *efficient solution* of the problem (S, f) when

$$(f(S) - f(x_0)) \cap (-K) = \{0\}\,.$$

We denote by $\mathrm{Eff}(S, f)$ the set of all efficient solutions of the problem (S, f). The image of the set $\mathrm{Eff}(S, f)$ under the function f is denoted by $\mathrm{Min}(S, f)$ and its elements are called *minimal points* of the set $f(S)$.

We recall also (see e.g. [6]) that a point $y_0 \in f(S)$ is said to be a *strictly minimal point* for problem $\mathrm{Min}(S, f)$, when for every $\epsilon > 0$ there exists $\delta > 0$, such that

$$(f(S) - y_0) \cap (\delta B - K) \subseteq \epsilon B.$$

We denote by $\mathrm{StMin}(S, f)$ the set of strictly minimal points for problem (S, f). Clearly $\mathrm{StMin}(S, f) \subseteq \mathrm{Min}(S, f)$. The next result gives a characterization of the strictly minimal points.

Proposition 2.1 *[6] Let $y_0 \in f(S)$. Then $y_0 \in \mathrm{StMin}(S, f)$ if and only if for every sequences $\{z_n\}$, $\{y_n\}$, with $\{z_n\} \subseteq f(S)$, $y_n \in z_n + K$ and $y_n \to y_0$, it holds $z_n \to y_0$.*

Now, we introduce the notion of *upper Hausdorff set-convergence.* Let $\{A_n\}$ be a sequence of subsets of X. We say that A_n upper converges to $A \subseteq X$ ($A_n \xrightarrow{H} A$) when $e(A_n, A) \to 0$, where $e(A_n, A) := \sup\limits_{a \in A_n} d(a, A)$ with $d(a, A) := \inf\limits_{b \in A} \|a - b\|$.

Besides the standard definitions of lower and upper semicontinuity, the following concept of continuity for set-valued maps will be used in this work. If $G : X \rightrightarrows Y$ is a set-valued map, we recall (see, e.g. [5]) that G is said to be *upper Hausdorff continuous* at x_0 when for every neighborhood V of 0 in Y there exists a neigborhood W of x_0 in X such that $G(x) \subseteq G(x_0) + V$ for every $x \in W \cap$ dom G,

3 Notions of well-posedness in vector optimization

Well-posedness is a classical topic in scalar optimization (see e.g. [12]). The first attempts to extend the notions of well-posedness to vector problems are in [3] and [23]. Recently, various notions of well-posedness in vector optimization have been introduced (see e.g. [4], [5], [19], [11], [16], [18], [26]). Among them, we focus on those notions which consider a fixed vector optimization problem, excluding the definitions that involve perturbations of the problem (the so called extended well-posedness [16], [17]). Thus, we consider the generalization to the vector case of the main ideas of Tykhonov's approach to well-posedness. In this section we gather and compare several definitions. Moreover, we list them in two separate classes, according to the "pointwise" or the "global" character of the considered notions. This classification is mainly motivated by a typical feature of vector optimization problems, i.e. even the image of the solution set is generally not a singleton.

3.1 Pointwise well-posedness

In this section we consider those notions of well-posedness, that do not take into account the whole solution set, but just a single point therein. We begin considering the following definition, given in [5].

Definition 3.1 *[5] The vector optimization problem (S, f) is called* B-well-posed *at $\bar{y} \in \mathrm{Min}(S, f)$ when the set-valued map $Q_{\bar{y}} : K \rightrightarrows S$ defined as*

$$Q_{\bar{y}}(\epsilon) := \{x \in S : f(x) \leq_K \bar{y} + \epsilon\},$$

is upper semicontinuous at $\epsilon = 0$ (observe that $Q_{\bar{y}}(0) = f^{-1}(\bar{y})$).

In [19] P. Loridan introduced a definition of well-posedness based on the notion of $\bar{y}$-minimizing sequence ($\bar{y} \in \mathrm{Min}(S, f)$), that is a sequence $\{x_n\} \subseteq S$ such that there exists a sequence $\{\epsilon_n\} \subseteq K$, $\epsilon_n \to 0$, with $f(x_n) \leq_K \bar{y} + \epsilon_n$.

Definition 3.2 *[19] The vector optimization problem (S, f) is called* L-well-posed *at $\bar{y} \in \mathrm{Min}(S, f)$ when every $\bar{y}$-minimizing sequence has a subsequence that converges to an element of $f^{-1}(\bar{y})$.*

Although Definition 3.1 is given through a continuity property of the map $Q_{\bar{y}}$, we can reformulate it in terms of $\bar{y}$-minimizing sequences.

Proposition 3.3 *Problem (S, f) is* B-well-posed *at $\bar{y} \in \mathrm{Min}(S, f)$ if and only if from every $\bar{y}$-minimizing sequence $\{x_n\} \subseteq S \backslash f^{-1}(\bar{y})$ one can extract a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to \bar{x} \in f^{-1}(\bar{y})$.*

Proof. Let (S, f) be a B-well-posed problem at $\bar{y} \in \mathrm{Min}(S, f)$ but there exists a $\bar{y}$-minimizing sequence $\{x_n\} \subseteq S \backslash f^{-1}(\bar{y})$ which admits no subsequence converging to an element of $f^{-1}(\bar{y})$. Since f is continuous, then $f^{-1}(\bar{y})$ is a closed set. Hence there exists an open set $W \subseteq X$ such that $f^{-1}(\bar{y}) \subseteq W$ and $x_n \notin W$. Since $x_n \in Q_{\bar{y}}(\epsilon_n)$ and $\epsilon_n \to 0$, we contradict the upper semicontinuity of the map $Q_{\bar{y}}(\epsilon)$ at $\epsilon = 0$.

Conversely, assume that $\{x_n\} \subseteq S$ is a $\bar{y}$-minimizing sequence. If problem (S, f) is not B-well-posed at $\bar{y} \in \mathrm{Min}(S, f)$, there exists an open set

$W \subseteq X$ with $f^{-1}(\bar{y}) \subseteq W$ and a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \notin W$, a contradiction. ■

As in the scalar case (see, e.g. [12]), a definition of vector well-posedness can be introduced considering the diameter of the level sets of the function f.

Definition 3.4 *[11] The vector optimization problem* (S, f) *is called* DH-well-posed *at* $\bar{x} \in \mathrm{Eff}(S, f)$ *when*

$$\inf_{\alpha>0} \mathrm{diam} L(\bar{x}, k, \alpha) = 0, \qquad \text{for each } k \in K,$$

where $L(x, k, \alpha) = \{x \in S : f(x) \leq_K f(\bar{x}) + \alpha k\}$.

We observe that if problem (S, f) is DH-well-posed at $\bar{x} \in \mathrm{Eff}(S, f)$, then $f^{-1}(f(\bar{x}))$ is a singleton.

Definition 3.5 *[18] The vector optimization problem* (S, f) *is called* H-well-posed *at* $\bar{x} \in \mathrm{Eff}(S, f)$ *when* $x_n \to \bar{x}$ *for any sequence* $\{x_n\} \subseteq S$ *such that* $f(x_n) \to f(\bar{x})$.

In the sequel we investigate the relations among the various notions of pointwise well-posedness.

Proposition 3.6 *[19] If the vector optimization problem* (S, f) *is L-well-posed at* $\bar{y} \in \mathrm{Min}(S, f)$*, then it is B-well-posed at* $\bar{y}$.

It is easy to see that in general the converse implication does not hold. Nevertheless, L-well-posedness at $\bar{y}$ trivially coincides with B-well-posedness at $\bar{y}$, whenever $f^{-1}(\bar{y})$ is compact.

The next result compares DH-well-posedness and L-well-posedness.

Proposition 3.7 *If problem* (S, f) *is DH-well-posed at* $\bar{x} \in \mathrm{Eff}(S, f)$*, then* (S, f) *is L-well-posed at* $\bar{y} = f(\bar{x})$.

Proof. Recall that, since (S, f) is DH-well-posed then $f^{-1}(\bar{y}) = \{\bar{x}\}$ and ab absurdo assume the existence of a sequence $\{x_n\} \subseteq S$ and a positive number γ such that $f(x_n) \leq_K \bar{y} + \epsilon_n$ and $x_n \notin \bar{x} + \gamma B$. Now, let $k \in \mathrm{int}\, K$ be fixed. For every $\alpha > 0$, there exists $n(\alpha)$ such that for $n > n(\alpha)$, $\epsilon_n \leq_K \alpha k$. Then, for $n > n(\alpha)$, x_n belongs to $L(\bar{x}, k, \alpha)$, which contradicts DH-well-posedness.

■

If we assume that $f^{-1}(\bar{y})$ is a singleton, then we can reverse the previous proposition.

Huang in [18] shows that DH-well-posedness at $\bar{y}$ implies H-well-posedness at $\bar{y}$. Example 2.1 in [18] shows that the converse implication does not hold. The same example shows that H-well-posedness at $\bar{y}$ implies neither L-well-posedness at $\bar{y}$ nor B-well-posedness at $\bar{y}$.

The next scheme summarizes the main relations among *pointwise well-posedness* notions.

$$\begin{array}{ccc} \text{L-well-posedness} & \Longleftarrow & \text{DH-well-posedness} \\ \Downarrow & & \Downarrow \\ \text{B-well-posedness} & & \text{H-well-posedness} \end{array}$$

3.2 Global well-posedness

In this subsection we list and compare those notions of vector well-posedness that consider the solution set as a whole. We underline that all the following definitions extend to the vector case the notion of scalar well-setness (see [7], [8]), which is more appropriate when the solution set is not compact.

Definition 3.8 *[5] The vector optimization problem (S, f) is called* B-well-posed *when* $\mathrm{Min}(S, f) \neq \emptyset$ *and the set-valued map $Q : K \rightrightarrows S$ defined as*

$$Q(\epsilon) := \bigcup_{y \in \mathrm{Min}(S,f)} \{x \in S : f(x) \leq_K y + \epsilon\},$$

is upper semicontinuous at $\epsilon = 0$.

In [5], the author introduces also another notion of well-posedness, where the continuity requirement on the map Q is weakened.

Definition 3.9 *The vector optimization problem (S, f) is called* weakly B-well-posed *when* $\mathrm{Min}(S, f) \neq \emptyset$ *and the set-valued map $Q : K \rightrightarrows S$ is upper Hausdorff continuous at $\epsilon = 0$.*

It trivially follows from definitions that B-well-posedness implies weak B-well-posedness. The converse does not hold, as shown by Example 3.1 in [26].

The previous definitions can be reformulated (see [5], Propositions 4.5 and 4.6) in terms of behaviour of appropriate minimizing sequences. The sequence $\{x_n\} \subseteq S$ is a B-minimizing sequence of the problem (S, f) when for each n there exist $\epsilon_n \in K$ and $y_n \in$Min(S, f) such that $f(x_n) \leq_K y_n + \epsilon_n$, where $\epsilon_n \rightarrow 0$. Now Definition 3.8 is equivalent to require that every B-minimizing sequence $\{x_n\} \subseteq S \backslash \text{Eff}(S, f)$ contains a subsequence converging to an element of Eff(S, f). Further, Definition 3.9 is equivalent to require that the distance of every minimizing sequence from the solution set Eff(S, f) converges to zero.

Another approach to global well-posedness is based on the behaviour of minimizing sequences of sets i.e. of sequences $\{A_n\} \subseteq S$ such that $f(A_n) \xrightarrow{H} \text{Min}(S, f)$.

Definition 3.10 *[26] The vector optimization problem (S, f) is called* M-well-posed *when every minimizing sequences of sets $\{A_n\}$ is such that $A_n \xrightarrow{H} \text{Eff}(S, f)$.*

Also the last definition can be rephrased in terms of sequences. It holds

$$\text{dist}(x_n, \text{Eff}(S, f)) \rightarrow 0$$

whenever $\text{dist}(f(x_n), \text{Min}(S, f)) \rightarrow 0,\ \{x_n\} \subseteq S$.

Now we compare the previous definitions of global well-posedness. The next result states that weak B-well-posedness is stronger than M-well-posedness.

Proposition 3.11 *[26] If (S, f) is weakly B-well-posed then it is M-well-posed.*

The converse implication holds only under additional assumptions, as shown by Example 3.14 below.

Proposition 3.12 *If (S, f) is M-well-posed and $\forall \epsilon > 0$, $\exists \delta > 0$, such that $f(S) \cap (\text{Min}(S, f) + \delta B - K) \subseteq \epsilon B$, then it is also weakly B-well-posed.*

Proof. The proof is analogous to that of Theorem 4.5 in [26]. ■

Remark 3.13 *The condition* $\forall\epsilon > 0$, $\exists\delta > 0$, *such that* $f(S) \cap (\mathrm{Min}(S, f) + \delta B - K) \subseteq \epsilon B$, *implies* $\mathrm{Min}(S, f) \subseteq \mathrm{StMin}(S, f)$. *In [26] (see Theorem 4.5) the previous proposition was stated under this weaker condition. In a private communication to the authors, C. Zălinescu pointed out some misleading argument in the proof. In fact, the result presented in [26] does not hold as the following example shows.*

Example 3.14 *Let* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *be the identity function,* $K = \mathbb{R}^2_+$ *and* $S = \{(x, y) \in \mathbb{R}^2 : x \geq 0 \vee y \geq 0 \vee y \geq -x - 1\}$. *We have* $\mathrm{Min}(S, f) = \mathrm{StMin}(S, f) = \{(x, y) \in \mathbb{R}^2 : y = -x - 1, -1 < x < 0\}$. *The condition of the previous proposition does not hold, and problem* (S, f) *is M-well-posed, but not weakly B-well-posed.*

The next scheme summarizes the main relations among *global well-posedness* notions.

B-well-posedness $\Longrightarrow$ weak B-well-posedness $\Longrightarrow$ M-well-posedness

We close this section recalling a result which links the pointwise and global approaches to well-posedness of vector optimization problems.

Proposition 3.15 *[5] Assume that* $\mathrm{Min}(S, f)$ *is compact. If, for each* $y \in \mathrm{Min}(S, f)$, *problem* (S, f) *is B-well-posed at* y, *then* (S, f) *is B-well-posed.*

We wish to observe that the compactness assumption on $\mathrm{Min}(S, f)$ is essential in order that the previous proposition holds, as the following example shows.

Example 3.16 *Consider the function* $f : \mathbb{R}^2 \to \mathbb{R}^2$, *defined as* $f(x, y) = (x, -x + ye^{-x})$ *and let* $S = K = \mathbb{R}^2_+$. *Here* $\mathrm{Min}(S, f) = \{(x, -x) : x \geq 0\}$. *Then, problem* (S, f) *is B-well-posed at* y, *for every* $y \in \mathrm{Min}(S, f)$, *but it is not B-well-posed. In fact the sequence* $\{(n, n)\} \subseteq S$ *is B-minimizing and* $(n, n) \notin \mathrm{Eff}(S, f)$, *but there exists no subsequence converging to an element of* $\mathrm{Eff}(S, f)$.

The following example shows that the implication of Proposition 3.15 cannot be reversed.

Example 3.17 *Consider the function* $f : \mathbb{R} \to \mathbb{R}^2$, $f(x) = (xe^{-x}, -xe^{-x})$ *and let* $S = \mathbb{R}_+$ *and* $K = \mathbb{R}^2_+$. *Then problem* (S, f) *is B-well-posed, but it is not B-well-posed at* $\bar{y} = (0, 0)$.

3.3 Other notions of well-posedness

Though we focus our attention on the notions that generalize Tykhonov-type well-posedness, we wish to quote some different extensions of well-posedness to the vector case. In [11] and [18] the authors introduce also some notions of well-posedness for vector optimization problems, that take into account also minimizing sequences out of the feasible set S. These approaches extend to the vector case the ideas of Levitin-Polyak well-posedness for scalar optimization problems.
In [16], [17] the author generalizes to the vector case the notion of extended well-posedness introduced in [28] for scalar optimization. In this definition also perturbations of the objective function are considered.

4 Well-posedness and strict efficiency

The result of this section shows that most notions of vector well-posedness impose some restrictions on the set $\mathrm{Min}(S, f)$. Indeed, every minimal point reveals to be a strictly minimal point. This property is typical of the vector case and shows that most of the vector well-posedness notions implicitly require stronger properties than the simple good behaviour of minimizing sequences.

Theorem 4.1 *If problem (S, f) is B-well-posed, then* $\mathrm{Min}(S, f) = \mathrm{StMin}(S, f)$.

Proof. We have only to prove that $\mathrm{Min}(S, f) \subseteq \mathrm{StMin}(S, f)$. By contradiction suppose that there exists a point

$$\bar{y} \in \mathrm{Min}(S, f) \backslash \mathrm{StMin}(S, f).$$

From Proposition 2.1 we get the existence of two sequences $\{z_n\}$ and $\{y_n\}$, with $y_n \to \bar{y}$, $z_n \in f(S)$, $y_n \in z_n + K$, a sequence $\{n_s\} \subseteq \mathbb{N}$ and a positive number r, such that $z_{n_s} \notin B_r(\bar{y})$. We have

$$z_{n_s} \leq_K y_{n_s} = \bar{y} + b_s,$$

where $b_s \to 0$. If we consider an element $e \in \operatorname{int} K$, there exists a sequence of positive real numbers λ_s, such that $\lambda_s \to 0$ and $\lambda_s e + b_s \in K$. Hence $z_{n_s} \leq_K y_{n_s} = \bar{y} + b_s \leq_K \bar{y} + \lambda_s e + b_s$. We consider now a sequence $\{x_s\}$ such that $x_s \in f^{-1}(z_{n_s})$. Clearly $\{x_s\}$ is a $\bar{y}$-minimizing sequence and since problem (S, f) is B-well-posed, then x_s converges to a point $\bar{x} \in \mathrm{Eff}(S, f)$. Since f is continuous, then $z_{n_s} = f(x_s) \to f(\bar{x})$ and we get $f(\bar{x}) \leq_K \bar{y}$, which implies $f(\bar{x}) = \bar{y}$. The last equality contradicts to $z_{n_s} \notin B_r(\bar{y})$ and completes the proof. ■

We observe that the previous result does not hold when we consider M-well-posedness, as one can show with easy examples. This is not surprising since this notion concerns only the interplay, through the function f, between the behaviour of minimizing sequences in Y and X.

Remark 4.2 *If problem (S, f) is B-well-posed at $\bar{y} \in \mathrm{Min}(S, f)$, then we can prove analogously that $\bar{y} \in \mathrm{StMin}(S, f)$.*

5 Well-posedness of scalarized problems

In this section we study the relationships between the well-posedness of vector optimization problems and the well-posedness of associated scalar problems, in order to find a scalarization that preserves well-posedness.

Consider the (scalar) minimization problem (S, h)

$$\min\ h(x), \quad x \in S$$

where $h : X \to \mathbb{R}$ and denote by $\mathrm{argmin}(S, h)$ the solution set of problem (S, h). We recall that a sequence $\{x_n\} \subseteq S$ is minimizing for problem (S, h), when $f(x_n) \to \inf_S f$.

Now we recall some notions of well-posedness for scalar optimization problems (see, e.g. [12]).

Definition 5.1 *Problem (S, h) is said to be* well-posed in the generalized sense *when every minimizing sequence has some subsequence converging to an element of* $\mathrm{argmin}(S, h)$.

When $\mathrm{argmin}(S, h)$ is a singleton, the previous definition reduces to the classical notion of Tykhonov well-posedness.
We recall also a generalization of the above mentioned notion.

Definition 5.2 *[7] Problem (S, h) is said to be* well-set *when every minimizing sequence $\{x_n\}$ contained in $S \backslash \mathrm{argmin}(S, h)$ has a cluster point in* $\mathrm{argmin}(S, h)$.

Among various scalarization procedures known in the literature, we consider the one based on the so called "oriented distance" function from the point y to the set A, introduced in [15] in the framework of nonsmooth scalar optimization. Later, it has been used to obtain a scalarization of a vector optimization problem in [10],[1],[25].

Definition 5.3 *For a set* $A \subseteq Y$*, the oriented distance function* $\Delta_A : Y \to \mathbb{R} \cup \{\pm\infty\}$ *is defined as*

$$\Delta_A(y) = d_A(y) - d_{Y \setminus A}(y),$$

where $d_A(y) = \inf_{x \in A} \|y - x\|$.

The main properties of function Δ_A are gathered in the following proposition (see [27]).

Proposition 5.4 *i) If* $A \neq \emptyset$ *and* $A \neq Y$ *then* Δ_A *is real valued;*

ii) Δ_A *is 1-Lipschitzian;*

iii) $\Delta_A(y) < 0$ *for every* $y \in \operatorname{int} A$*,* $\Delta_A(y) = 0$ *for every* $y \in \partial A$ *and* $\Delta_A(y) > 0$ *for every* $y \in \operatorname{int} A^c$*;*

iv) if A *is closed, then it holds* $A = \{y : \Delta_A(y) \leq 0\}$*;*

v) if A *is convex, then* Δ_A *is convex;*

vi) if A *is a cone, then* Δ_A *is positively homogeneouos;*

vii) if A *is a closed convex cone, then* Δ_A *is nonincreasing with respect to the ordering relation induced on* Y*, i.e. the following is true: if* $y_1, y_2 \in Y$ *then*

$$y_1 - y_2 \in A \quad \Longrightarrow \quad \Delta_A(y_1) \leq \Delta_A(y_2)$$

If A *has nonempty interior, then*

$$y_1 - y_2 \in \operatorname{int} A \quad \Longrightarrow \quad \Delta_A(y_1) < \Delta_A(y_2)$$

The scalar problem associated with the vector problem (S, f) is the following

$$\min \Delta_{-K}(f(x) - p), \quad x \in S$$

where $p \in Y$. In the sequel we denote this problem by (S, Δ_{-K}). The relations of the solutions of this problem with those of problem (S, f) are investigated in [24], [25], [27] and [14]. For the convenience of the reader, we quote the characterization of minimal points and strictly minimal points in the image space.

Theorem 5.5 *[27] Let $\bar{y} \in f(S)$.*

i) $\bar{y} \in \mathrm{Min}(S, f)$ if and only if $\bar{y}$ is the unique solution of the scalar optimization problem

$$\min\ \Delta_{-K}(y - \bar{y}), \quad y \in f(S).$$

ii) $\bar{y} \in \mathrm{StMin}(S, f)$ if and only if the previous problem is Tykhonov well-posed.

The next results provide some equivalences between well-posedness of the vector problem (S, f) and of the related scalar problem (S, Δ_{-K}).

Theorem 5.6 *Let $\bar{y} \in \mathrm{Min}(S, f)$. Problem (S, Δ_{-K}) with $p = \bar{y}$ is well-set if and only if problem (S, f) is B-well-posed at $\bar{y}$.*

Proof. Let $\bar{y} \in \mathrm{Min}(S, f)$ and assume (S, Δ_{-K}) is well-set. Let $\{x_n\}$ be a $\bar{y}$-minimizing sequence, i.e. such that $\{x_n\} \subseteq S \backslash f^{-1}(\bar{y})$ and $f(x_n) \leq_k \bar{y} + \epsilon_n$ with $\epsilon_n \in K$ and $\epsilon_n \to 0$. Hence, $f(x_n) = \bar{y} + \epsilon_n - k_n$, with $k_n \in K$. We have

$$\Delta_{-K}(f(x_n) - \bar{y}) =$$

$$\Delta_{-K}(\epsilon_n - k_n) \leq \Delta_{-K}(\epsilon_n) + \Delta_{-K}(-k_n),$$

where the last inequality follows since Δ_{-K} is subadditive (see Proposition 5.4). We get $\Delta_{-K}(-k_n) \leq 0$ for every n and $\Delta_{-K}(\epsilon_n) \to 0$ as $n \to +\infty$. Since $\bar{y} \in \mathrm{Min}(S, f)$, by Theorem 5.5 it holds $\Delta_{-K}(\epsilon_n - k_n) \geq 0$ for every n. Then, $\Delta_{-K}(\epsilon_n - k_n) \to 0$ as $n \to +\infty$ and hence x_n is a minimizing sequence for the scalar problem (S, Δ_{-K}) with $p = \bar{y}$. It follows that there

exists a subsequence x_{n_k} converging to a point $\bar{x} \in \operatorname{argmin}(S, \Delta_{-K})$. Since, by Theorem 5.5 $\Delta_{-K}(y - \bar{y})$ has $\bar{y}$ as unique minimizer on $f(S)$, we have $f(\bar{x}) = \bar{y}$. Then problem (S, f) is B-well-posed at $\bar{y}$.
Conversely, let $\bar{x} \in \operatorname{Eff}(S, f)$ such that $f(\bar{x}) = \bar{y}$ and assume that (S, f) is B-well-posed at $\bar{y}$. By Theorem 5.5 $\bar{x}$ solves problem (S, Δ_{-K}) with $p = \bar{y}$. Let $\{x_n\}$ be such that $\{x_n\} \subseteq S$, $y_n = f(x_n) \neq f(\bar{x})$ and $\Delta_{-K}(y_n - \bar{y}) \to 0$. By Theorem 4.1 we have that $\bar{y} \in \operatorname{StMin}(S, f)$. Theorem 5.5, ensures that problem

$$\min \Delta_{-K}(y - \bar{y}), \quad y \in f(S)$$

is Tykhonov well-posed and hence $y_n \to \bar{y}$. It follows that $y_n = \bar{y} + \alpha_n$, where $\alpha_n \in Y$ and $\alpha_n \to 0$. Since $\operatorname{int} K \neq \emptyset$, we can write $\alpha_n = \epsilon_n - \gamma_n$, with $\epsilon_n,\ \gamma_n \in K$ and so $y_n = \bar{y} - \gamma_n + \epsilon_n$. By Proposition 2.1, we have $\bar{y} - \gamma_n \to \bar{y}$, that is $\gamma_n \to 0$ and this implies $\epsilon_n \to 0$. Hence x_n is a $\bar{y}$-minimizing sequence for problem (S, f) and since this problem is B-well-posed at $\bar{y}$, there exists a subsequence $\{x_{n_k}\} \to \tilde{x} \in f^{-1}(\bar{y})$. It follows that $\tilde{x}$ solves problem (S, Δ_{-K}) with $p = \bar{y}$ and the proof is complete.
■

The following corollary follows from Corollary 1.4 in [7].

Corollary 5.7 *Problem (S, f) is L-well-posed at $\bar{y} \in \operatorname{Min}(S, f)$ if and only if problem (S, Δ_{-K}), with $p = \bar{y}$ is well-posed in the generalized sense.*

When problem (S, f) is DH-well-posed at $\bar{x} \in \operatorname{Eff}(S, f)$, observing that $f^{-1}(f(\bar{x})) = \{\bar{x}\}$, we get the following corollary.

Corollary 5.8 *Let $\bar{x} \in \operatorname{Eff}(S, f)$. Then problem (S, f) is DH-well-posed at $\bar{x}$ if and only if problem (S, Δ_{-K}) with $p = f(\bar{x})$ is Tykhonov well-posed.*

Remark 5.9 (Linear scalarization). *Linear scalarization for vector optimization problems is widely used mainly in the convex case (see e.g. [22]). Anyway, for linear scalarization, the parallelism between vector and scalar well-posedness notions does not hold even in the convex case as already observed in [23]. For instance, consider problem (S, f) where $S = \{(x, y) \in \mathbb{R}^2 : y \geq -x\}$, f is the identity function, $K = \mathbb{R}^2_+$ and the point $(0, 0) \in \operatorname{Min}(S, f)$. Then, this problem is DH-well-posed at $(0, 0)$ (hence L-well-posed and B-well-posed at $(0, 0)$), but there is no linearly scalarized problem with $(0, 0)$ as a solution, which is Tykhonov well-posed.*

6 Well-posedness of quasiconvex vector optimization problems

In this section we study the well-posedness properties of quasiconvex vector optimization problems.

Definition 6.1 *Let S be a convex set. A function $f : X \to Y$ is said to be* K-quasiconvex *on S, when for $y \in Y$, the level sets*

$$\{x \in S : f(x) \leq_K y\}$$

are convex.

From now on we assume that X and Y are finite dimensional euclidean spaces and we denote the inner product in Y by $\langle \cdot, \cdot \rangle$.

Proposition 6.2 *[14] Let $K^+ = \{v \in Y : \langle v, y \rangle \geq 0, \ \forall y \in K\}$ be the positive polar of the cone K, consider a compact base G for K^+ and define the function*

$$\tilde{g}(y) = \max_{\xi \in G} \langle \xi, y - \bar{y} \rangle,$$

where $\bar{y} \in Y$. Then there exist positive constants α and β, such that for every $y \in Y$ it holds

$$\alpha \tilde{g}(y) \leq \Delta_{-K}(y - \bar{y}) \leq \beta \tilde{g}(y)\,.$$

Remark 6.3 *From the previous result, recalling Theorem 5.5, it follows easily that $\bar{y} \in \mathrm{Min}(S, f)$ if and only if $\bar{y}$ is the unique minimizer of function $\tilde{g}(y)$ on $f(S)$.*

Lemma 6.4 *i) Let $e \in \operatorname{int} K$. Then the set $H \cap K^+$, where H is the hyperplane $H = \{y \in Y : \langle e, y \rangle = 1\}$, is a compact base for K^+.*

ii) Let $G = H \cap K^+$ and consider the function $g(x) = \tilde{g}(f(x))$. If f is K-quasiconvex on the convex set S, then $g(x)$ is quasiconvex on S.

Proof. The first statement is well-known (see e.g. [22]).
To prove the second statement, let $\alpha \in \mathbb{R}$ and consider the level set

$$\{x \in S : g(x) \leq \alpha\} = \{x \in S : \max_{\xi \in G} \langle \xi, f(x) - \bar{y} \rangle \leq \alpha\} =$$

$$\{x \in S : \max_{\xi \in G} \langle \xi, f(x) - \bar{y} \rangle \leq \alpha \max_{\xi \in G} \langle \xi, e \rangle\} =$$

$$\{x \in S : \max_{\xi \in G} \langle \xi, f(x) - \bar{y} - \alpha e \rangle \leq 0\} = \{x \in S : f(x) - \bar{y} - \alpha e \in -K\} =$$

$$\{x \in S : f(x) \leq_K \bar{y} + \alpha e\}.$$

Since f is K-quasiconvex, the last set is convex and the lemma is proved. ■

Remark 6.5 *From the inequalities presented in Proposition 6.2 it follows easily that the well-posedness in the generalized sense of function $g(x)$ is equivalent to that of function $\Delta_{-K}(f(x) - \bar{y})$, where $\bar{y} \in \mathrm{Min}(S, f)$.*

Proposition 6.6 *Let f be K-quasiconvex on the convex set S, let $\bar{y} \in \mathrm{Min}(S, f)$ and assume that $f^{-1}(\bar{y})$ is compact. Then problem (S, f) is L-well-posed at $\bar{y}$.*

Proof. From Remark 6.3 it follows that the set $\mathrm{argmin}(S, g)$ is compact. Furhermore, it is easy to see that if f is continuous, then g is continuous too. Since a continuous quasiconvex function with compact set of minimizers is well-posed in the generalized sense (Theorem 2.1 in [9]), the proof follows from Lemma 6.4 and Corollary 5.7. ■
It straightforward to see that, under the assumptions of the previous proposition, problem (S, f) is also B-well-posed at $\bar{y}$.

The next proposition concerns the global well-posedness of vector quasiconvex optimization problems. It follows easily from the previous proposition and Proposition 3.15.

Proposition 6.7 *Let f be K-quasiconvex on the convex set S. For every $\bar{y} \in \mathrm{Min}(S, f)$ assume that $f^{-1}(\bar{y})$ is compact and suppose that $\mathrm{Min}\,(S, f)$ is compact. Then problem (S, f) is B-well-posed.*

We underline that the last proposition improves Theorem 5.5 in [26], where the authors deal with a convex problem and assume that $\mathrm{Eff}(S, f)$ is compact.

Propositions 6.6 and 6.7 can be further generalized, with similar proofs, extending them to K- quasiconnected functions, which can be wieved as an extension of the notion of scalar quasiconnected function (see e.g. [2]). If S is an arcwise connected set, a function $f : X \to Y$ is said to be K-quasiconnected on S when for every $x_1,\ x_2 \in S$, there exixts a continuous path $\gamma(t; x_1, x_2) : [0,1] \to S$, with $\gamma(0; x_1, x_2) = x_1$ and $\gamma(1; x_1, x_2) = x_2$, such that the following implication holds

$$f(x_1), f(x_2) \leq_K y \ \text{ with } \ y \in Y \Rightarrow f(\gamma(t; x_1, x_2)) \leq_K y \ \text{ for every } \ t \in [0,1].$$

We observe that when K-quasiconnected functions are considered, the previous results generalize those presented in [9] for scalar optimization problems.

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