

I.Ginchev, A. Guerraggio, M. Rocca

Isolated minimizers, proper
efficiency and stability for $C^{0,1}$
constrained vector optimization
problems

2004/9



UNIVERSITÀ DELL'INSUBRIA
FACOLTÀ DI ECONOMIA

<http://eco.uninsubria.it>

In questi quaderni vengono pubblicati i lavori dei docenti della Facoltà di Economia dell'Università dell'Insubria. La pubblicazione di contributi di altri studiosi, che abbiano un rapporto didattico o scientifico stabile con la Facoltà, può essere proposta da un professore della Facoltà, dopo che il contributo sia stato discusso pubblicamente. Il nome del proponente è riportato in nota all'articolo. I punti di vista espressi nei quaderni della Facoltà di Economia riflettono unicamente le opinioni degli autori, e non rispecchiano necessariamente quelli della Facoltà di Economia dell'Università dell'Insubria.

These Working papers collect the work of the Faculty of Economics of the University of Insubria. The publication of work by other Authors can be proposed by a member of the Faculty, provided that the paper has been presented in public. The name of the proposer is reported in a footnote. The views expressed in the Working papers reflect the opinions of the Authors only, and not necessarily the ones of the Economics Faculty of the University of Insubria.

© Copyright I. Ginchev, A. Guerraggio, M. Rocca
Printed in Italy in March 2004
Università degli Studi dell'Insubria
Via Ravasi 2, 21100 Varese, Italy

All rights reserved. No part of this paper may be reproduced in any form without permission of the Author.

Isolated minimizers, proper efficiency and stability for $C^{0,1}$ constrained vector optimization problems

Ivan Ginchev

Technical University of Varna

Department of Mathematics

9010 Varna, Bulgaria

ginchev@ms3.tu-varna.acad.bg

Angelo Guerraggio

University of Insubria

Department of Economics

21100 Varese, Italy

aguerraggio@eco.uninsubria.it

Matteo Rocca

University of Insubria

Department of Economics

21100 Varese, Italy

mrocca@eco.uninsubria.it

Abstract

In this paper we consider the vector optimization problem $\min_C f(x), g(x) \in -K$, where $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}^p$ are $C^{0,1}$ functions and $C \subset \mathbb{R}^m$ and $K \subset \mathbb{R}^p$ are closed convex cones. We give several notions of solutions (efficiency concepts), among them the notion of a properly efficient point (p -minimizer) of order k and the notion of an isolated minimizer of order k . We show that each isolated minimizer of order $k \geq 1$ is a p -minimizer of order k . The possible reversal of this statement in the case $k = 1$ is the main subject of the investigation. For this study we apply some first order necessary and sufficient conditions in terms of Dini derivatives. We show that the given optimality conditions are important to solve the posed problem, and a satisfactory solution leads to two approaches toward efficiency concepts, called respectively sense I and sense II concepts. Relations between sense I and sense II isolated minimizers and p -minimizers are obtained. In particular, we are concerned in the stability properties of the p -minimizers and the isolated minimizers. By stability, we mean that they still remain the same type of solutions under small perturbations of the problem data. We show that the p -minimizers are stable under perturbations of the cones, while the isolated minimizers are stable under perturbations both of the cones and the functions in the data set. Further, we show that the sense I concepts are stable under perturbations of the objective data, while the sense II concepts are stable under perturbations both of the objective and the constraints.

2000 Mathematics Subject Classification: 90C29, 49J52.

Keywords: Vector optimization, Locally Lipschitz data, Properly efficient points, Isolated minimizers, Optimality conditions, Stability.

1 Introduction

In this paper we consider the vector optimization problem

$$\min_C f(x), \quad g(x) \in -K, \quad (1)$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $g : \mathbb{R}^n \rightarrow \mathbb{R}^p$. Here n , m and p are positive integers and $C \subset \mathbb{R}^n$ and $K \subset \mathbb{R}^p$ are closed convex cones. It generalizes from scalar to vector optimization the classical Fritz John problem. There are different types of solutions of problem (1). Usually the solutions are called points of efficiency. We prefer, as in scalar optimization, to call them minimizers.

Let us underline, that for simplicity we assume that the functions f and g in (1) are defined on the whole space \mathbb{R}^n . The results of the paper remain true, if as usually in optimization, these functions are supposed to be defined on an open set in \mathbb{R}^n .

The deep connection between vector and scalar optimality concepts has always been stressed. Recall for instance that in the convex case a well known approach is the linear scalarization of the vector problem. In nonconvex problems several ad hoc scalarization techniques have been used. In this paper we consider a particular kind of scalarization which makes use of the so called "oriented distance" from a point to a set. It has been shown (see e.g. [11, 12, 27]) that more restrictive definitions of minimality for the considered scalarized problem correspond to more restrictive notions of efficiency.

In this work we are interested in the links between isolated minimizers (see e.g. [2]) of the scalarized problem and properly efficient points of the constrained problem (1). We will assume that f and g are of class $C^{0,1}$, i.e. locally Lipschitz functions. For such functions we apply some first-order necessary and sufficient optimality conditions to clarify the relations between these concepts.

While exploring the links between isolated minimizers and proper efficiency of the constrained problem (1), we give a new notion of proper efficiency that we will call proper efficiency in sense II (while we will refer to the classical notion of proper efficiency as proper efficiency in sense I). We show that this kind of proper efficiency implies stability of the solution with respect to the constraints and, under some regularity assumption, reveals to be a stronger notion than the classical one.

The outline of the paper is the following. Section 2 is devoted to some preliminary concepts. Here we recall several kinds of solutions of a vector optimization problem, among them the notions of properly efficient points (p -minimizers) and isolated minimizers of order k . Here we introduce the oriented distance function and also the main results linking scalar and vector optimality concepts. Section 3 generalizes the notion of a p -minimizer to a p -minimizer of order k . It starts the investigations of the links between isolated minimizers and proper efficiency by showing (Theorems 3.1 and 3.2) that each

isolated minimizer is a p -minimizer. The possible reversal of this statement in the case $k = 1$ is the main subject of investigation in the paper. In Section 4 with reference to $C^{0,1}$ functions, we recall some first order necessary and sufficient conditions in terms of Dini derivatives given in [12]. Section 5 discusses a reversal of Theorem 3.2, shows that the given optimality conditions are important to solve this problem, and that a satisfactory solution leads to two approaches toward the efficiency concepts, called respectively sense I and sense II concepts. The relation between the sense I and sense II isolated minimizers and p -minimizers is investigated. In Section 6 we investigate the stability properties of the p -minimizers and the isolated minimizers. We show that sense I and sense II concepts concern differences in the stability behaviour of the solutions. We mean by stability of a solution x^0 , that x^0 remains the same type of solution under small perturbations of the problem data. We show that the p -minimizers are stable under perturbations of the cones, while the isolated minimizers are stable under perturbations both of the cones and the functions in the data set. Further, we show that the sense I concepts are stable under perturbations of the objective data, while the sense II concepts are stable under perturbations both of the objective and the constraints.

2 Vector optimality concepts and scalar characterizations

For the Euclidean norm and the scalar product in the considered finite-dimensional spaces we write $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$. The open unit ball is denoted by B . From the context it should be clear to exactly which spaces these notations are applied. Considering Euclidean spaces for simplicity, let us mention, that the considerations can be raised immediately to finite dimensional real Banach spaces with the convention that $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ stands then respectively for the Banach norm and the dual pairing.

There are different concepts of solution of problem (1). In any case a solution x^0 should be a feasible point, i.e. $g(x) \in -K$ (equivalently $x \in g^{-1}(-K)$), which is assumed in the following definitions. The definitions presented are given in a local sense. We omit this specification in the text.

Definition 2.1. i) *The feasible point x^0 is said to be weakly efficient (efficient) point, if there is a neighbourhood U of x^0 , such that if $x \in U \cap g^{-1}(-K)$ then $f(x) - f(x^0) \notin -\text{int } C$ (respectively $f(x) - f(x^0) \notin -(C \setminus \{0\})$).*

ii) *The feasible point x^0 is said to be properly efficient if there exists closed (but not necessarily convex) cone $\tilde{C} \subset \mathbb{R}^n$, such that $C \setminus \{0\} \subset \text{int } \tilde{C}$ and there exists a neighbourhood U of x^0 , such that if $x \in U \cap g^{-1}(-K)$, then $f(x) - f(x^0) \notin -\text{int } \tilde{C}$.*

It is worth to mention, that Definition 2.1 works also assuming for C and K only that they are closed cones, dropping the assumption of their convexity. Though then the

cones do not introduce partial order in their spaces, vector optimization with nonconvex cones appears in the literature. However for us the convexity assumption is important, since lately we make use of it introducing the oriented distance function and its analytical representation.

In this paper the weakly efficient, the efficient and the properly efficient points of problem (1) are called respectively w -minimizers, e -minimizers and p -minimizers. The following chain of implications is known:

$$p\text{-minimizer} \Rightarrow e\text{-minimizer} \Rightarrow w\text{-minimizer} .$$

In virtue of Definition 2.1 a p -minimizer can be defined in the following way: The feasible point x^0 is said to be properly efficient point for the constrained problem (1) if there exists a closed cone \tilde{C} , such that $C \setminus \{0\} \subset \text{int } \tilde{C}$ and x^0 is weakly efficient point for the problem $\min_{\tilde{C}} f(x), g(x) \in -K$. The equivalence of the two definitions holds true if this optimization problem does not implicitly assume, that is as a result of the general assumptions on the considered vector optimization problems, some additional properties of the involved cones. In our case such an implicit assumption is the convexity of \tilde{C} . The definition of a p -minimizer with the additional assumption of \tilde{C} closed convex cone remains equivalent to Definition 2.1 ii) only in the case if the cone C is pointed. Often in the literature the considerations of the vector optimization problems are restricted to the case of C pointed closed convex cone with nonempty interior. The demand nonempty interior makes the weakly efficient points an interesting object (if the cone C has an empty interior, then each feasible point x^0 is a weakly efficient point for the constrained problem (1)). For the investigations in this paper both conditions C pointed and with nonempty interior are too restrictive (they need not be satisfied in the considered later problem (13)) and we prefer to get rid of them at the very beginning.

We give also the following definition.

Definition 2.2. *The feasible point x^0 is said a strong e -minimizer if there exists a neighborhood U of x^0 , such that $f(x) - f(x^0) \notin -C$, for $x \in U \setminus \{x^0\} \cap g^{-1}(-K)$.*

Obviously, every strong e -minimizer is e -minimizer.

The unconstrained problem

$$\min_C f(x), \quad x \in \mathbb{R}^n, \tag{2}$$

is a particular case of problem (1) and the defined notions of optimality are obviously related also to this problem.

For the cone $M \subset \mathbb{R}^k$ its positive polar cone M' is defined by $M' = \{\zeta \in \mathbb{R}^k \mid \langle \zeta, \phi \rangle \geq 0 \text{ for all } \phi \in M\}$. The cone M' is closed and convex. It is well known that $M'' := (M')' =$

$\text{cl co } M$, see e. g. Rockafellar [26, Chapter III, § 15]. In particular for the closed convex cone M we have $M' = \{\zeta \in \mathbb{R}^k \mid \langle \zeta, \phi \rangle \geq 0 \text{ for all } \phi \in M\}$ and $M = M'' = \{\phi \in \mathbb{R}^k \mid \langle \zeta, \phi \rangle \geq 0 \text{ for all } \zeta \in M'\}$.

If $\phi \in -\text{cl co } M$, then $\langle \zeta, \phi \rangle \leq 0$ for all $\zeta \in M'$. We set $M'(\phi) = \{\zeta \in M' \mid \langle \zeta, \phi \rangle = 0\}$. Then $M'(\phi)$ is a closed convex cone and $M'(\phi) \subset M'$. Consequently its positive polar cone $M(\phi) = (M'(\phi))'$ is a closed convex cone, $M \subset M(\phi)$ and its positive polar cone satisfies $(M(\phi))' = M'(\phi)$. In this paper we apply this notation for $M = K$ and $\phi = g(x^0)$. Then we write for short $K'(x^0)$ instead of $K'(g(x^0))$ (and call this cone the index set of problem (1) at x^0) and $K(x^0)$ instead of $K(g(x^0))$. We find this abbreviation convenient and not ambiguous, since further this is the unique case, in which we make use of the cones $M'(\phi)$ and $M(\phi)$.

A relation of the vector optimization problem (1) to some scalar optimization problem can be obtained in terms of positive polar cones.

Proposition 2.1 ([12]). *Define*

$$\varphi(x) = \max \{ \langle \xi, f(x) - f(x^0) \rangle \mid \xi \in C', \|\xi\| = 1 \}. \quad (3)$$

The feasible point $x^0 \in \mathbb{R}^n$ is a w -minimizer for problem (1), if and only if x^0 is a minimizer for the scalar problem

$$\min \varphi(x), \quad g(x) \in -K. \quad (4)$$

Proposition 2.2 ([12]). *The feasible point x^0 is a strong e -minimizer of problem (1) if and only if x^0 is a strong minimizer of problem (4), i.e. if and only if there exists a neighborhood U of x^0 , such that $\varphi(x) - \varphi(x^0) > 0$ for all $x \in (U \setminus \{x^0\}) \cap g^{-1}(-K)$.*

Recall that the feasible point x^0 is said to be an isolated minimizer of order $k > 0$ of problem (4) when there exists a constant $A > 0$ such that $\varphi(x) \geq \varphi(x^0) + A \|x - x^0\|^k$ for all $x \in U \cap g^{-1}(-K)$. The concept of an isolated minimizer for scalar problems has been popularized by Auslender [2]. It is natural to introduce a similar concept of optimality for the vector problem (1).

Definition 2.3. *We say that the feasible point x^0 is an isolated minimizer of order k for the vector problem (1) if it is an isolated minimizer of order k for the scalar problem (4).*

To interpret geometrically the property that x^0 is a minimizer of problem (1) of certain type we introduce the so called oriented distance. Given a set $A \subset \mathbb{R}^k$, then the distance from $y \in \mathbb{R}^k$ to A is given by $d(y, A) = \inf \{ \|a - y\| \mid a \in A \}$. This definition works also for $A = \emptyset$ putting $d(y, \emptyset) = \inf \emptyset = +\infty$. The oriented distance from y to A is defined by

$D(y, A) = d(y, A) - d(y, \mathbb{R}^k \setminus A)$. This definition in the case $A = \emptyset$ gives $D(y, A) = +\infty$ and in the case $A = \mathbb{R}^k$ it gives $D(y, A) = -\infty$.

The function D is introduced in Hiriart-Urruty [16, 17] and is used later in Ciligot-Travain [9], Amahroq, Taa [1], Miglierina [23], Miglierina, Molho [24]. Zaffaroni [27] gives different notions of efficiency and uses the function D for their scalarization and comparison. Ginchev, Hoffmann [13] use the oriented distance to study approximation of set-valued functions by single-valued ones and in case of a convex set A show the representation $D(y, A) = \sup_{\|\xi\|=1} (\inf_{a \in A} \langle \xi, a \rangle - \langle \xi, y \rangle)$. From this representation, if C is a convex cone and taking into account

$$\inf_{a \in C} \langle \xi, a \rangle = \begin{cases} 0 & , \quad \xi \in C', \\ -\infty & , \quad \xi \notin C', \end{cases}$$

we get easily $D(y, -C) = \sup_{\|\xi\|=1, \xi \in C'} \langle \xi, y \rangle$. Turn attention, that this formula works also in the case of the improper cones $C = \{0\}$ (then $D(y, -C) = \sup_{\|\xi\|=1} \langle \xi, y \rangle = \|y\|$) and $C = \mathbb{R}^m$ (then $D(y, -C) = \sup_{\xi \in \emptyset} \langle \xi, y \rangle = -\infty$).

In particular the function φ in (4) is expressed by $\varphi(x) = D(f(x) - f(x^0), -C)$. Proposition 2.1 is easily reformulated in terms of the oriented distance, namely:

$$\begin{aligned} x^0 \text{ } w\text{-minimizer} & \Leftrightarrow D(f(x) - f(x^0), -C) \geq 0 \text{ for } x \in U \cap g^{-1}(-K), \\ x^0 \text{ strong } e\text{-minimizer} & \Leftrightarrow D(f(x) - f(x^0), -C) > 0 \text{ for } x \in (U \setminus \{x^0\}) \cap g^{-1}(-K). \end{aligned}$$

The definition of the isolated minimizers gives

$$\begin{aligned} x^0 \text{ isolated minimizer of order } k & \Leftrightarrow \\ D(f(x) - f(x^0), -C) \geq A \|x - x^0\|^k & \text{ for } x \in U \cap g^{-1}(-K). \end{aligned}$$

We see that the isolated minimizers are strong e -minimizers. In the next section we explore the links between isolated minimizers and p -minimizers.

Remark 2.1. *In the important case $C = \mathbb{R}_+^n$ it can be shown (see [11, 12]) that statements like the ones in Propositions 2.1 and 2.2 remain true if the function φ is substituted by*

$$\varphi_0(x) = \max_{1 \leq i \leq n} (f_i(x) - f_i(x^0)). \quad (5)$$

In fact, there exist constants $\alpha, \beta > 0$ such that $\alpha \varphi(x) \leq \varphi_0(x) \leq \beta \varphi(x)$.

We conclude the section with a comment concerning the made in the beginning of the paper assumption that the cones C and K are closed and convex. In principle dealing with the vector optimization problem (1) one is inclined to accept that the cone C should introduce a partial order in virtue of $y^1 \leq_C y^2 \Leftrightarrow y^2 - y^1 \in C$ and for the sake of transitivity of the partial order it is necessary that C should be convex. However Definitions 2.1 and

2.2 do not involve convexity assumptions for C and K , hence convexity is not essential for the concepts of efficiency. Convexity of C is however essential when applying dual concepts, in particular polar cones. It is important both for validity of Proposition 2.2 and for the representation $\varphi(x) = D(f(x) - f(x^0), -C)$. For similar reasons the convexity of K is important for the validity of the formulated further Theorem 4.1, which plays an important role in the present research.

3 Isolated minimizers and proper efficiency

Applying the introduced in the previous section oriented distance we can generalize the concept of proper efficiency. For a given cone C and given $k \geq 1$ and $a > 0$ we define the set

$$C^k(a) = \{y \in \mathbb{R}^m \mid D(y, C) \leq a \|y\|^k\}.$$

In the case $k = 1$ we write for short $C(a)$ instead of $C^1(a)$. Let us underline, that $C(a)$ is a cone containing C .

Definition 3.1. *We say that the feasible point x^0 is a properly efficient point of order $k \geq 1$ (called also p -minimizer of order k) for problem (1) if there exists a neighbourhood U of x^0 and a constant $a > 0$ such that if $x \in U \cap g^{-1}(-K)$ then $f(x) - f(x^0) \notin -\text{int } C^k(a)$.*

The above definition cannot be extended mechanically for the case $0 < k < 1$. Indeed, then for arbitrary $a > 0$ and all sufficiently small $\|y\|$ we would have $D(y, C) \leq \|y\| \leq a\|y\|^k$. Therefore assuming f is continuous, which has place when f is $C^{0,1}$, and x sufficiently close to x^0 , the inclusion $f(x) - f(x^0) \notin -\text{int } C^k(a)$ could not have place.

The definition of a p -minimizer of order k can be introduced also in another way. Denote by \hat{y} the orthogonal projection of y on C . Then we define the set $\hat{C}^k(a) = \{y \in \mathbb{R}^m \mid D(y, C) \leq a \|\hat{y}\|^k\}$. Consequently, we say that the feasible point x^0 is a properly efficient of order k (for short, p -minimizer of order k) if there exists a neighbourhood U of x^0 and a constant $a > 0$ such that if $x \in U \cap g^{-1}(-K)$ then $f(x) - f(x^0) \notin -\text{int } \hat{C}^k(a)$. It is easy to show, that this new definition has sense in general for $k > 0$, and for $k \geq 1$ is equivalent to Definition 3.1. However, under the assumption of locally Lipschitz data, which is the case in this paper, p -minimizers of order $k \in (0, 1)$ do not exist at all. For this reason we confine in the sequel to the case $k \geq 1$ and work with Definition 3.1.

Proposition 3.1. *The point x^0 is a p -minimizer for problem (1) if and only if it is a p -minimizer of order 1.*

Proof. If x^0 is a p -minimizer of order 1 then x^0 satisfies Definition 2.1 with respect to the cone $\tilde{C} = C(a)$, hence x^0 is a p -minimizer.

Conversely, let x^0 be a p -minimizer and \tilde{C} is the cone from Definition 2.1. Since the set $F = \{y \in C \mid \|y\| = 1\}$ is compact and disjoint from the closed set $\mathbb{R}^n \setminus \tilde{C}$, therefore $a := \text{dist}(F, \mathbb{R}^n \setminus \text{int } \tilde{C}) > 0$. Now obviously $C(a) \subset \tilde{C}$. Since $x \in U \cap g^{-1}(-K)$ implies $f(x) - f(x^0) \notin -\text{int } \tilde{C}$ and $-\text{int } \tilde{C} \supset -\text{int } \tilde{C}(a)$, we get $f(x) - f(x^0) \notin -\text{int } C(a)$. Therefore x^0 is a p -minimizer of order 1. \square

Proposition 3.1 shows that the given in Definition 3.1 notion of proper efficiency of order $k \geq 1$ generalizes the usual notion of proper efficiency. This is a justification of the proposed name. The given in Definition 3.1 condition can be expressed in different ways and on this basis we get different equivalent definitions of proper efficiency of order k . The next Proposition 3.2 submits such a proposal.

Proposition 3.2. *The feasible point x^0 is a p -minimizer of order k for problem (1) if and only if there exist a neighbourhood U of x^0 and a constant $a > 0$ such that for all $\varepsilon > 0$ and all $x \in U \cap g^{-1}(-K)$ satisfying $\|f(x) - f(x^0)\| \geq \varepsilon$ it holds $D(f(x) - f(x^0), -C) \geq a\varepsilon^k$.*

Proof. Let x^0 be a p -minimizer of order k . Then there exist a neighbourhood U of x^0 and a constant $a > 0$ such that for all $x \in U \cap g^{-1}(-K)$ it holds $f(x) - f(x^0) \notin -\text{int } C^k(a)$. With the account of the definition of $C^k(a)$ the latter gives that $D(f(x) - f(x^0), -C) \geq a\|f(x) - f(x^0)\|^k$. Then $\|f(x) - f(x^0)\| \geq \varepsilon$ gives $D(f(x) - f(x^0), -C) \geq a\varepsilon^k$.

Conversely, let x^0 satisfy the given condition. In particular the inequality $\|f(x) - f(x^0)\| \geq \varepsilon$ is satisfied if we fix $x \in U \cap g^{-1}(-K)$ in advance for $\varepsilon = \|f(x) - f(x^0)\|$. The latter inequality gives that $D(f(x) - f(x^0), -C) \geq a\|f(x) - f(x^0)\|^k$ which can be rephrased as $f(x) - f(x^0) \notin -\text{int } C^k(a)$. \square

The condition in Proposition 3.2 is convenient as a tool to check whether a given point x^0 is a p -minimizer of order k , see below the comments on Example 3.2. Let us however underline, that the same condition can be expressed in the equivalent form given in the next proposition.

Proposition 3.3. *The feasible point x^0 is a p -minimizer of order $k \geq 1$ for problem (1) if and only if there exist a neighbourhood U of x^0 and a constant $a > 0$ such that for all $\varepsilon > 0$ it holds*

$$(f(U \cap g^{-1}(-K)) - f(x^0)) \cap (a\varepsilon^k B - C) \subset \varepsilon B. \quad (6)$$

Proof. Let x^0 be p -minimizer of order k and let the neighbourhood U of x^0 and the constant $a > 0$ are that from Proposition 3.2. We show that (6) holds for all $\varepsilon > 0$. Assume in the contrary, that there exists $x \in U \cap g^{-1}(-K)$ such that $f(x) - f(x^0) \in a\varepsilon^k B - C$, or equivalently $D(f(x) - f(x^0), -C) < a\varepsilon^k$, but $f(x) - f(x^0) \notin \varepsilon B$, or equivalently $\|f(x) - f(x^0)\| \geq \varepsilon$. The latter inequality according to Proposition 3.2 implies $D(f(x) - f(x^0), -C) \geq a\varepsilon^k$, a contradiction.

Let now for x^0 there exist a neighbourhood U and a constant $a > 0$ for which (6) holds. We show that also the condition in Proposition 3.2 is satisfied. Assume in the contrary, that there exists $\varepsilon > 0$ and $x \in U \cap g^{-1}(-K)$ satisfying $\|f(x) - f(x^0)\| \geq \varepsilon$, but $D(f(x) - f(x^0), -C) < a\varepsilon^k$. This means that $f(x) - f(x^0)$ belongs to the left hand side of (6) but not to the right hand side, a contradiction. \square

To the best of our knowledge the proposed here definition of proper efficiency of order $k \geq 1$ is a new one. Let us however mention that Zălinescu [28] gives two definitions of proper efficiency, one of which applies a particular case of formula (6). For this reason [28] gives some prerequisite for the notion of p -minimizer of order k . Bednarczuk [5] calls the efficient points based on this definition strictly efficient points, and this name has been used also in [11]. From Proposition 3.3 it follows that the p -minimizers are strictly efficient points.

One can apply Definition 3.1 also to define p minimizers of order $k \in (0, 1)$, but for this purpose the definition of the set $C^k(a)$ needs first to be modified. In this paper we do not discuss this problem.

In the case when f is $C^{0,1}$ function, that is locally Lipschitz, the following relation between isolated minimizers of order k and p -minimizers of order k holds.

Theorem 3.1. *Let f be of class $C^{0,1}$. If a point x^0 is an isolated minimizer of order $k \geq 1$ for problem (1) then x^0 is a p -minimizer of order k .*

Proof. Assume in the contrary, that x^0 is an isolated minimizer of order k but not p -minimizer of order k . Let f be Lipschitz with constant L in $x^0 + r \operatorname{cl} B$. Take sequences $\delta_\nu \rightarrow 0+$ and $\varepsilon_\nu \rightarrow 0+$. Consider the sets $C^k(\varepsilon_\nu)$. From the assumption that x^0 is not a p -minimizer of order k it follows that there exists a sequence of feasible points $x^\nu \in (x^0 + \delta_\nu B) \cap g^{-1}(-K)$ such that $f(x^\nu) - f(x^0) \in -\operatorname{int} C^k(\varepsilon_\nu)$, and in particular $f(x^\nu) - f(x^0) \neq 0$. From the definition of $C^k(\varepsilon_\nu)$ we get

$$D(f(x^\nu) - f(x^0), -C) \leq \varepsilon_\nu \|f(x^\nu) - f(x^0)\|^k \leq \varepsilon_\nu L^k \|x^\nu - x^0\|^k$$

which contradicts to x^0 isolated minimizer of order k . \square

We formulate separately the particular case obtained by Theorem 3.1 for $k = 1$.

Theorem 3.2. *Let f be of class $C^{0,1}$. If x^0 is isolated minimizer of first order for problem (1) then x^0 is a p -minimizer.*

In the sequel we consider only isolated minimizers of first order and for this reason we will say occasionally just isolated minimizers instead of isolated minimizers of first order. Similarly, we consider also only p -minimizers of first order, which as we know are

just p -minimizers. Obviously, the first-order results obtained further in the paper admit generalizations to arbitrary order k .

The next Examples 3.1 and 3.2 show respectively that the Lipschitz assumption in Theorem 3.2 cannot be dropped and the result of Theorem 3.2 in general cannot be converted. As for the used notations, let us say that we prefer to denote the fixed value of the variable x by x^0 when x is vector-valued (then x_i^0 stands for the i -th coordinate of x^0) and x_0 when x is real-valued.

Example 3.1. Let $f : \mathbb{R} \rightarrow \mathbb{R}^2$, $g : \mathbb{R} \rightarrow \mathbb{R}$, be defined as $f(x) = (\sqrt{|x|}, -\sqrt[4]{|x|})$ and $g(x) = x$. Let $C = \mathbb{R}_+^2$ and $K = \mathbb{R}_+$. The point $x_0 = 0$ is an isolated minimizer of first order, but not a p -minimizer for problem (1).

From $f(x) = f(-x)$ we see that the condition $g(x) \equiv x \leq 0$ does not introduce changes on the efficiency properties of $x_0 = 0$ for the constrained problem (1) in comparison with the unconstrained problem (2). It is obvious from the definition that x_0 is not a p -minimizer. The function φ_0 in (5) is $\varphi_0(x) = \max(\sqrt{|x|}, -\sqrt[4]{|x|}) = \sqrt{|x|}$. Since $\varphi_0(x) \equiv \sqrt{|x|} \geq |x|$ for $|x| < 1$, the point x_0 is an isolated minimizer of first order. Thus, the conclusion of Theorem 3.2 does not hold, but obviously also f is not $C^{0,1}$.

Example 3.2. Let $f : \mathbb{R} \rightarrow \mathbb{R}^2$, $g : \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f(x) = (x^2, -x^2)$ and $g(x) = x$. Let $C = \mathbb{R}_+^2$ and $K = \mathbb{R}_+$. Hence, f and g are of class $C^{0,1}$, $x_0 = 0$ is a p -minimizer, but x_0 is not an isolated minimizer of first order.

As an illustration of an application of Proposition 3.2 we observe that in this example $D(f(x) - f(x_0), -C) = D((x^2, -x^2), -\mathbb{R}_+^2) = x^2 = \frac{1}{\sqrt{2}} \|(x^2, -x^2)\| = \frac{1}{\sqrt{2}} \|f(x) - f(x_0)\|$.

Therefore $f(x) - f(x_0) \notin -\text{int } C(1/\sqrt{2})$, whence x_0 is a p -minimizer. On the other hand x_0 is not an isolated minimizer of first order for (1), since x_0 is not an isolated minimizer of first order for the scalar problem $\varphi_0(x) = x^2 \rightarrow \min, x \leq 0$.

4 Dini derivatives and first-order optimality conditions

Problem (1) has been investigated in [12] under the hypothesis that f and g are of class $C^{0,1}$, i.e. locally Lipschitz. The authors obtained optimality conditions in terms of the first-order Dini directional derivative.

Given a $C^{0,1}$ function $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^k$ we define the Dini directional derivative (we use to say just Dini derivative) $\Phi'_u(x^0)$ of Φ at x^0 in direction $u \in \mathbb{R}^n$, as the set of the cluster points of $(1/t)(\Phi(x^0 + tu) - \Phi(x^0))$ as $t \rightarrow 0+$, that is as the Kuratowski limit

$$\Phi'_u(x^0) = \text{Limsup}_{t \rightarrow 0+} \frac{1}{t} (\Phi(x^0 + tu) - \Phi(x^0)) .$$

If Φ is Fréchet differentiable at x^0 then the Dini derivative is a singleton, coincides with the usual directional derivative and can be expressed in terms of the Fréchet derivative $\Phi'(x^0)$ (called sometimes the Jacobian of Φ at x^0) by

$$\Phi'_u(x^0) = \lim_{t \rightarrow 0^+} \frac{1}{t} (\Phi(x^0 + tu) - \Phi(x^0)) = \Phi'(x^0)u.$$

In connection with problem (1) we deal with the Dini directional derivative of the function $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^{m+p}$, $\Phi(x) = (f(x), g(x))$ and then we use to write $\Phi'_u(x^0) = (f, g)'_u(x^0)$. If at least one of the derivatives $f'_u(x^0)$ and $g'_u(x^0)$ is a singleton, then $(f, g)'_u(x^0) = (f'_u(x^0), g'_u(x^0))$. Let us turn attention that always $(f, g)'_u(x^0) \subset f'_u(x^0) \times g'_u(x^0)$, but in general these two sets do not coincide.

The first-order necessary and sufficient optimality conditions given in the next theorem will be useful in clarifying the links between isolated minimizers and p -minimizers. In its formulation the following constrained qualification is used being a generalization for $C^{0,1}$ constraints of the well known Kuhn-Tucker constrained qualification, compare with Mangasarian [22, page 102].

$$\mathcal{Q}_{0,1}(x^0) : \quad \begin{array}{l} \text{If } g(x^0) \in -K \text{ and } \frac{1}{t_k} (g(x^0 + t_k u^0) - g(x^0)) \rightarrow z^0 \in -K(x^0) \\ \text{then } \exists u^k \rightarrow u^0 : \exists k_0 \in \mathbb{N} : \forall k > k_0 : g(x^0 + t_k u^k) \in -K. \end{array}$$

Theorem 4.1 ([12]). *Let f, g be $C^{0,1}$ functions.*

(Necessary Conditions) *Let x^0 be a w -minimizer of problem (1). Then for each $u \in \mathbb{R}^n$ the following condition is satisfied:*

$$\mathcal{N}'_{0,1} : \quad \begin{array}{l} \forall (y^0, z^0) \in (f, g)'_u(x^0) : \exists (\xi^0, \eta^0) \in C' \times K' : \\ (\xi^0, \eta^0) \neq (0, 0), \quad \langle \eta^0, g(x^0) \rangle = 0 \quad \text{and} \quad \langle \xi^0, y^0 \rangle + \langle \eta^0, z^0 \rangle \geq 0. \end{array}$$

(Sufficient Conditions) *Let $x^0 \in \mathbb{R}^n$ and suppose that for each $u \in \mathbb{R}^n \setminus \{0\}$ the following condition is satisfied:*

$$\mathcal{S}'_{0,1} : \quad \begin{array}{l} \forall (y^0, z^0) \in (f, g)'_u(x^0) : \exists (\xi^0, \eta^0) \in C' \times K' : \\ (\xi^0, \eta^0) \neq (0, 0), \quad \langle \eta^0, g(x^0) \rangle = 0 \quad \text{and} \quad \langle \xi^0, y^0 \rangle + \langle \eta^0, z^0 \rangle > 0. \end{array}$$

Then x^0 is an isolated minimizer of first order for problem (1).

Conversely, if x^0 is an isolated minimizer of first order for problem (1) and the constraint qualification $\mathcal{Q}_{0,1}(x^0)$ holds, then condition $\mathcal{S}'_{0,1}$ is satisfied.

Theorem 4.1 is valid and simplifies in an obvious manner when instead of (1) the unconstrained problem (2) is considered. Let us underline that then the reversal of the sufficient conditions does not require the use of constraint qualifications.

Theorem 4.2. (Necessary Conditions) *Let f be $C^{0,1}$ functions. Let x^0 be a w -minimizer of problem (2). Then for each $u \in \mathbb{R}^n$ the following condition is satisfied:*

$$\forall y^0 \in f'_u(x^0) : \exists \xi^0 \in C' : \xi^0 \neq 0 \text{ and } \langle \xi^0, y^0 \rangle \geq 0.$$

(Sufficient Conditions) Let $x^0 \in \mathbb{R}^n$ and suppose that for each $u \in \mathbb{R}^n \setminus \{0\}$ the following condition is satisfied:

$$\forall y^0 \in f'_u(x^0) : \exists \xi^0 \in C' : \xi^0 \neq 0 \text{ and } \langle \xi^0, y^0 \rangle > 0. \quad (7)$$

Then x^0 is an isolated minimizer of first order for problem (2). Conversely, if x^0 is an isolated minimizer of first order for problem (2) then condition (7) is satisfied.

As an obvious application of Theorem 4.1 and in some connection to p -minimizers we get the next Proposition 4.1.

Proposition 4.1. Let f and g be locally Lipschitz functions. If, for some pair $(\xi^0, \eta^0) \in (C' \times K'(x^0)) \setminus \{(0, 0)\}$, the feasible point x^0 is an isolated minimizer of first order for the scalar function

$$\varphi^0(x) = \langle \xi^0, f(x) - f(x^0) \rangle + \langle \eta^0, g(x) \rangle, \quad (8)$$

then x^0 is a p -minimizer of (1).

Proof. Let $u \in \mathbb{R}^n \setminus \{0\}$ and let $(y^0, z^0) \in (f, g)'_u(x^0)$. Hence, for some sequence $t_k \rightarrow 0+$, we have

$$y^0 = \lim_{k \rightarrow +\infty} \frac{f(x^0 + t_k u) - f(x^0)}{t_k}, \quad z^0 = \lim_{k \rightarrow +\infty} \frac{g(x^0 + t_k u) - g(x^0)}{t_k}.$$

Since x^0 is an isolated minimizer of first order for the scalar function (8), there exists a number $A > 0$, such that $\varphi^0(x^0 + t_k u) - \varphi^0(x^0) \geq A t_k$, whence

$$\langle \xi^0, \frac{1}{t_k} (f(x^0 + t_k u) - f(x^0)) \rangle + \langle \eta^0, \frac{1}{t_k} (g(x^0 + t_k u) - g(x^0)) \rangle \geq A > 0.$$

Passing to the limit we get $\langle \xi^0, y^0 \rangle + \langle \eta^0, z^0 \rangle \geq A > 0$. Now the Sufficient Condition in Theorem 4.1 gives that x^0 is an isolated minimizer of first order for problem (1), and according to Theorem 3.2 it is also p -minimizer. \square

5 Two approaches toward proper efficiency

After Theorem 3.2 it is natural to put the question, under what condition this theorem admits a reversal. In other words under what condition x^0 p -minimizer implies x^0 isolated minimizer of first order. Example 3.2 shows that in general such a reversal does not hold. To answer the posed question we consider first the unconstrained problem (2). Then a crucial role plays the property $0 \notin f'_u(x^0)$.

Theorem 5.1. Let f be a locally Lipschitz function and let x^0 be a p -minimizer for the unconstrained problem (2), which has the property $0 \notin f'_u(x^0)$ for all $u \in \mathbb{R}^n \setminus \{0\}$. Then x^0 is an isolated minimizer of first order for (2).

Proof. We prove separately the particular case, when C is a pointed cone, in order to demonstrate a smart application of Theorem 4.2. Thereafter we prove the general case.

The case of C pointed. According to the comments after Definition 2.1 we may assume that the cone \tilde{C} in this definition is closed convex, such that $\text{int } \tilde{C} \supset C \setminus \{0\}$ and x^0 is w -minimizer for the problem $\min_{\tilde{C}} f(x)$, $x \in \mathbb{R}^n$. According to the Necessary Conditions of Theorem 4.2, this means, that for each $u \in \mathbb{R}^n \setminus \{0\}$ and $y^0 \in f'_u(x^0)$, there exists $\tilde{\xi}^0 \in \tilde{C} \setminus \{0\}$, such that $\langle \tilde{\xi}^0, y^0 \rangle \geq 0$. This inequality, together with $y^0 \neq 0$ (implied by property $\mathcal{P}(x^0, u)$), shows that $y^0 \notin -\text{int } \tilde{C} \cup \{0\}$. Since $C \subset \text{int } \tilde{C} \cup \{0\}$, we see that $y^0 \notin -C$. This implies, that there exists $\xi^0 \in C'$, such that $\langle \xi^0, y^0 \rangle > 0$. According to the reversal of the Sufficient Conditions of Theorem 4.2, the point x^0 is an isolated minimizer of first order.

The general case. The general case assumes that the cone C is only closed and convex. Therefore Definition 2.1 of a p -minimizer demands that the cone \tilde{C} is only closed, and not necessarily convex.

Assume in the contrary, that x^0 is a p -minimizer for the unconstrained problem (2), but it is not an isolated minimizer of first order. Choose a monotone decreasing sequence $\varepsilon_k \rightarrow 0+$. From the assumption, there exist sequences $t_k \rightarrow 0+$ and u^k , $\|u^k\| = 1$, such that

$$D(f(x^0 + t_k u^k) - f(x^0), -C) = \max_{\xi \in \Gamma_{C'}} \langle \xi, f(x^0 + t_k u^k) - f(x^0) \rangle < \varepsilon_k t_k. \quad (9)$$

Here $\Gamma_{C'} = \{\xi \in C' \mid \|\xi\| = 1\}$. We may assume that $0 < t_k < r$ and f is Lipschitz with constant L in $x^0 + r \text{cl} B$. Passing to a subsequence, we may assume also that $u^k \rightarrow u^0$, $\|u^0\| = 1$, and that $y^0 = \lim_k y^{0,k}$, where $y^k = (1/t_k)(f(x^0 + t_k u^k) - f(x^0))$ and $y^{0,k} = (1/t_k)(f(x^0 + t_k u^0) - f(x^0))$. From the definition of the Dini derivative we have $y^0 \in f'_u(x^0)$ and from the made assumption $y^0 \neq 0$. We show that $y^k \rightarrow y^0$. This follows from the estimation

$$\|y^k - y^0\| \leq \frac{1}{t_k} \|f(x^0 + t_k u^k) - f(x^0 + t_k u^0)\| + \|y^{0,k} - y^0\| \leq L \|u^k - u^0\| + \|y^{0,k} - y^0\|.$$

Let now $\xi \in \Gamma_{C'}$. Then

$$\begin{aligned} \langle \xi, y^k \rangle &= \frac{1}{t_k} \langle \xi, f(x^0 + t_k u^k) - f(x^0) \rangle \leq \frac{1}{t_k} \max_{\xi \in \Gamma_{C'}} \langle \xi, f(x^0 + t_k u^k) - f(x^0) \rangle \\ &= \frac{1}{t_k} D(f(x^0 + t_k u^k) - f(x^0), -C) < \frac{1}{t_k} \varepsilon_k t_k = \varepsilon_k. \end{aligned}$$

Passing to a limit with $k \rightarrow \infty$ we get $\langle \xi, y^0 \rangle \leq 0$ for arbitrary $\xi \in \Gamma_{C'}$, whence $D(y^0, -C) = \max_{\xi \in \Gamma_{C'}} \langle \xi, y^0 \rangle \leq 0$.

On the other hand x^0 is a p -minimizer, which according to Proposition 3.1 means that x^0 is p -minimizer of order 1. Definition 3.1 gives now that there exists a constant $a > 0$,

such that for all sufficiently great k it holds

$$f(x^0 + t_k u^k) - f(x^0) \notin -\text{int } C^k(a) \Leftrightarrow \frac{1}{t_k} D(f(x^0 + t_k u^k) - f(x^0), -C) \geq a \left\| \frac{1}{t_k} (f(x^0 + t_k u^k) - f(x^0)) \right\|.$$

Applying the positive homogeneity of the oriented distance and passing to a limit with $k \rightarrow \infty$ we get

$$0 \geq D(y^0, -C) \geq a \|y^0\| > 0, \quad (10)$$

a contradiction, which shows that x^0 is an isolated minimizer. \square

Now we generalize Theorem 5.1 for the constrained problem.

Theorem 5.2. *Let f and g be $C^{0,1}$ functions and let x^0 be a p -minimizer for the constrained problem (1), which has the property*

$$(y^0, z^0) \in (f, g)'_u(x^0) \text{ and } z^0 \in -K(x^0) \text{ implies } y^0 \neq 0. \quad (11)$$

Then x^0 is an isolated minimizer of first order for (1).

Proof. Assume in the contrary, that x^0 is a p -minimizer for the constrained problem (1) but it is not an isolated minimizer of first order. Choose a monotone decreasing sequence $\varepsilon_k \rightarrow 0+$. From the assumption, there exist sequences $t_k \rightarrow 0+$ and u^k , $\|u^k\| = 1$, such that $g(x^0 + t_k u^k) \in -K$ and (9). We may assume that $0 < t_k < r$ and f and g are locally Lipschitz with constant L in $x^0 + r \text{ cl } B$. Passing to a subsequence we may assume also that $u^k \rightarrow u^0$, $\|u^0\| = 1$, and that $y^0 = \lim_k y^{0,k}$ and $z^0 = \lim_k z^{0,k}$. Here $y^k = (1/t_k)(f(x^0 + t_k u^k) - f(x^0))$, $y^{0,k} = (1/t_k)(f(x^0 + t_k u^k) - f(x^0))$. Similarly $z^k = (1/t_k)(g(x^0 + t_k u^k) - g(x^0))$, $z^{0,k} = (1/t_k)(g(x^0 + t_k u^k) - g(x^0))$. Obviously $(y^0, z^0) \in (f, g)'_{u^0}(x^0)$ and like in the general case proof of Theorem 5.1 we have $y^0 = \lim_k y^k$ and $z^0 = \lim_k z^k$. Further $z^0 \in -K(x^0)$, which is true, since $\eta \in K'(x^0)$ implies $\langle \eta, z^k \rangle = (1/t_k) \langle \eta, g(x^0 + t_k u^k) \rangle \leq 0$. Therefore condition (11) gives $y^0 \neq 0$. Repeating now the general case proof of Theorem 5.1 we get the contradictory chain of inequalities (10), which proves the thesis. \square

As we see from Theorems 5.1 and 5.2 the condition $0 \notin f'_u(x^0)$ plays an important role for the implication x^0 p -minimizer implies x^0 isolated minimizer of first order. However, as next Example 5.1 shows, in the constrained case, this condition is not necessary for this implication (while it is in the unconstrained case as Theorem 4.1 shows).

Example 5.1. *Consider the constrained problem (1) with $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = -x^2$, $C = \mathbb{R}_+$, $g : \mathbb{R} \rightarrow \mathbb{R}$, $g(x) = |x|$, $K = \mathbb{R}_+$. The point $x_0 = 0$ is the only feasible point and according to the definitions in Section 2 it is both a p -minimizer and an isolated minimizer of first order. The Dini derivative at x_0 in direction u is $(f, g)'_u(x_0) = (0, |u|)$.*

The proof of Theorem 5.1, the case of C pointed, convince us in the importance of the Sufficient conditions, when investigating the implication x^0 p -minimizer implies x^0 isolated minimizer of first order. These sufficient conditions “appear in an implicit form” in the proof of Theorem 5.1 (In the sense, that Theorem 4.1 in [12] applies similar reasonings in its proof). The sufficient condition $\mathcal{S}'_{0,1}$ in Theorem 4.1 involves in fact the condition

$$(0, 0) \notin (f, g)'_u(x^0) \text{ for all } u \in \mathbb{R}^n \setminus \{0\}. \quad (12)$$

Indeed, if $(y^0, z^0) = (0, 0) \in (f, g)'_u(x^0)$, then the strong inequality $\langle \xi^0, y^0 \rangle + \langle \eta^0, z^0 \rangle > 0$ for $(\xi^0, \eta^0) \in C' \times K'(x^0)$ cannot be satisfied. Therefore, it seems natural, for the investigated implication, instead of condition (11) from Theorem 5.2 to apply condition (12). The next example shows however, that the conclusion of Theorem 5.2 does not hold, when replacing condition (11) with condition (12).

Example 5.2. Consider problem (1), with $f : \mathbb{R} \rightarrow \mathbb{R}^2$, $f(x) = (x^2, -x^2)$, $C = \mathbb{R}_+^2$, $g : \mathbb{R} \rightarrow \mathbb{R}$, $g(x) = -|x|$, $K = \mathbb{R}_+$ and let $x_0 = 0$. For $u \in \mathbb{R} \setminus \{0\}$ we have $f'_u(x_0) = (0, 0)$ and $(f, g)'_u(x^0) = (0, 0; -|u|) \neq 0$. Therefore condition (12) has place, but (11) does not. Further $g(x_0) = 0$, whence we derive easily that $K(x_0) = \mathbb{R}_+$. The constrained qualification $\mathcal{Q}_{0,1}(x^0)$ is satisfied, since $g(x_0 + tu) = -t|u| \in -\mathbb{R}_+$ for every $u \in \mathbb{R}$ and $t > 0$. The point x_0 is a p -minimizer, but not an isolated minimizer of first order. Therefore, the conclusion of Theorem 5.2 has not place.

In virtue of Example 5.2, to obtain a result similar to Theorem 5.2 under condition (12) we need a new approach toward the concepts of an isolated minimizer and a p -minimizer. For this purpose, we relate to the constrained problem (1) and the feasible point x^0 , the unconstrained problem

$$\min_{C \times K(x^0)} (f(x), g(x)) \quad (13)$$

Definition 5.1. We say that x^0 is a p -minimizer of order k in sense II (or just p -minimizer in sense II, when $k = 1$) for the constrained problem (1) if x^0 is a p -minimizer of order k for the unconstrained problem (13).

Similarly, we say that x^0 is an isolated minimizer of order k in sense II for the constrained problem (1) if x^0 is an isolated minimizer of order k for the unconstrained problem (13).

We will preserve the names for the concepts used so far, but sometimes they will be referred to as sense I concepts, saying e.g. p -minimizer in sense I, instead of just p -minimizer.

As an immediate application of Theorem 5.1 we get the following result.

Theorem 5.3. *Let f and g be $C^{0,1}$ functions and let x^0 be a p -minimizer in sense II for the constrained problem (1), which has property (12). Then x^0 is an isolated minimizer of first order in sense II for (1).*

Next we show, that under the hypotheses of Theorem 5.3 we can get the conclusion that x^0 is an isolated minimizer of first order in sense I. We state also some relations between isolated minimizers of first order in sense I and II, and similarly between p -minimizers in sense I and II.

Theorem 5.4. *Let f and g be $C^{0,1}$ functions and let x^0 be a p -minimizer in sense II for the constrained problem (1), which has property (12). Then x^0 is an isolated minimizer of first order in sense I for (1), and hence, x^0 is also a p -minimizer in sense I.*

Proof. According to Theorem 5.3 x^0 is an isolated minimizer of first order for the unconstrained problem (13). The reversal of the Sufficient conditions part of Theorem 4.2 gives a condition, which coincides with the sufficient condition $\mathcal{S}'_{0,1}$ of Theorem 4.1, whence x^0 is an isolated minimizer in sense I for the constrained problem (1). Theorem 3.2 gives now, that x^0 is also a p -minimizer in sense I for (1). \square

Thus, within the set of points satisfying (12), the set of the p -minimizers in sense II is a subset of the p -minimizers in sense I. The reversal does not hold. In fact, the following reasoning shows, that in Example 5.2 the point x_0 is a p -minimizer in sense I, but it is not a p -minimizer in sense II. Now for the corresponding problem (13) we have

$$(f, g) : \mathbb{R} \rightarrow \mathbb{R}^3, \quad (f(x), g(x)) = (x^2, -x^2, -|x|)$$

and $C \times K(x_0) = \mathbb{R}_+^2 \times \mathbb{R}_+ = \mathbb{R}_+^3$. Each point $x \in \mathbb{R}$ is feasible and the function φ_0 from (5) is $\varphi_0(x) = \max(x^2, -x^2, -|x|) = x^2$, whence x_0 is an isolated minimizer of order 2 in sense II, but it is not an isolated minimizer of first order in sense II. Therefore, according to Theorem 5.3, in spite that x_0 is a p -minimizer in sense I, it is not a p -minimizer in sense II (the assumption that x_0 is a p -minimizer in sense II would imply that x_0 is an isolated minimizer in sense II).

Let us now make some comparison of Theorems 5.2 and 5.4. In spite that condition (12) is more general than condition (11), Theorem 5.4 does not imply Theorem 5.2. Indeed, the assumption in Theorem 5.4 is that x^0 is a p -minimizer in sense II, which does not imply the more general assumption that x^0 is a p -minimizer in sense I.

Next we compare the isolated minimizers in sense I and II.

Theorem 5.5. *Let f and g be $C^{0,1}$ functions. If x^0 is an isolated minimizer of first order in sense II for the constrained problem (1), then x^0 is an isolated minimizer of first order in sense I for (1). If the constraint qualification $\mathcal{Q}_{0,1}(x^0)$ holds, then also the converse is true.*

Proof. Let x^0 be an isolated minimizer of first order in sense II. The reversal of the Sufficient conditions part of Theorem 4.2 gives the sufficient condition $\mathcal{S}'_{0,1}$ of Theorem 4.1, whence x^0 is an isolated minimizer in sense I.

Conversely, let x^0 be an isolated minimizer of first order in sense I. Under the constraint qualification $\mathcal{Q}_{0,1}(x^0)$, we can apply the reversal of the Sufficient conditions part of Theorem 4.1), getting condition $\mathcal{S}'_{0,1}$, which is identical with the sufficient conditions of Theorem 4.2 applied to problem (13), whence x^0 is an isolated minimizer in sense II. \square

We conclude the paper with the following remark. The comparison of the p -minimizers and the isolated minimizers of first order has been a motivation to introduce besides the p -minimizers (in sense I) defined in Section 2, also the p -minimizers in sense II. From the literature it is known, that usually some stability properties are appropriate both to p -minimizers (see e.g. [6], [25] and [24]) and to the isolated minimizers (see e.g. [2]). An interesting question is to compare the type of stability which is appropriate respectively to the sense I and sense II concepts. The authors' intention is to show in a separate paper, that sense I concepts obey stability with respect to the objective data, while sense II concepts obey stability with respect both to the objective data and the constraint data.

6 Proper efficiency, isolated minimizers and stability

In this section we discuss stability properties of properly efficient points and isolated minimizers for the constrained vector optimization problem (1).

Consider the following example.

Example 6.1. *In the unconstrained problem (2) with $f : \mathbb{R} \rightarrow \mathbb{R}^2$, $f(x) = (x^2, x)$, and $C = \mathbb{R}_+^2$ the point $x_0 = 0$ is a strong minimizer but not a p -minimizer. At the same time each point $x < 0$ is a p -minimizer.*

Following this example Podinovskiy, Nogin [25] explain the ‘‘anomaly’’ of the solution x_0 in the following words:

Moving from $x_0 = 0$ with $f(x_0) = (0, 0)$ to an arbitrary close efficient point $x < 0$ with $f(x) = (x^2, x)$ we gain an advantage of first order with respect to the second criterion f_2 as a result of only second order loss with respect to the first criterion f_1 . If f_1 is not considered to be essentially more important than f_2 , it is natural to accept some increase of f_1 admitting an one order less decrease of f_2 . The considered example shows, that it is sensible to particularize the efficient solutions not obeying such anomalies. The first definition of such a solution, called properly efficient, was given by H. Kuhn

and A. Tucker [20]. It has been formulated for the differentiable case and has been connected with special regularity qualifications allowing to get necessary optimality conditions. For the general case the definition of proper efficiency has been proposed by A. Geoffrion [10]. His notion of proper efficiency however uses essentially the coordinate character of the partial order and does not admit a straightforward generalization to vector optimization with respect to more general partial order. This limitation is overcome by J. Borwein [7] who defines proper efficiency with respect to order given by cones.

This quotation convinces us that the notion of proper efficiency has undergone some development. The applied in this paper Definition 2.1 of a p -minimizer is closer to Henig [15]. For this reason the given there notion of a properly efficient point, called later also p -minimizer in sense I, is referred to as the classical definition of a p -minimizer, in order to be distinguished from the introduced in the previous section p -minimizer in sense II. However, we do not risk to claim, that our approach is identical to any approach that one can find in the literature. Let us underline, that nowadays it does not exist a unique commonly accepted definition of a properly efficient point. The comments after Definition 2.1 show that the content of the notion of proper efficiency may vary or may be modified depending on the task it is used to. Survey on proper efficiency and different approaches to this concept give Guerraggio, Molho, Zaffaroni [14].

The properly efficient points obey some stability properties, which have been a subject of investigation since short after the notion appeared in the literature, see e.g. Benson, Morin [6]. However stability can be understood in different ways. The quotation above convinces us that the efficient but non-properly efficient points x^0 are “instable” in the following sense. If an arbitrary neighbourhood of x_0 contains properly efficient points, then the latter are preferable as efficiency estimations. Conversely, the properly efficient points are “stable” in sense that they do not obey this kind of “instability”. However the commonly accepted idea of stability is as a type of relation between the solutions of the given problem and a perturbed problem, more precisely that small perturbations of the problem data lead to a small change of the solution, in which the type of the solution is preserved. We may speak on well posedness too. With such an understanding Miglierina, Molho [24] prove certain type of stability for properly efficient points. Their approach however concerns efficient boundaries of sets and it is not appropriate for comparison of different notions of proper efficiency and stability for constrained problems. Some peculiarities concerning stability, when constrained optimization problems are investigated, consider Balayadi, Sonntag, Zălinescu [3]. Their approach however relates to usual and not to vector optimization.

In this section we discuss stability of the solutions of the constrained vector optimiza-

tion problem (1). Our point of view to stability is even stronger than the mentioned above. We are concerned with solutions x^0 of the constrained problem (1), which remain to be the same type of solutions under small perturbations of the initial data. The closed links between p -minimizers and isolated minimizers claim that the stability properties of the isolated minimizers deserve also a detailed study. In fact, for scalar problems the stability properties of the isolated minimizers are shown in Auslender [2]. In the present paper the generalization is twofold. First, we generalize stability properties from scalar to vector problems, and second, we pay a special attention to constrained problems. The latter introduce some new features. Concerning constrained problems, let us recall, that we introduced two type of optimality concepts, referred to as sense I and sense II concepts. Our task is to distinguish between the stability properties of the sense I and sense II concepts. We show, that p -minimizers are stable under perturbations of the ordering cones (see below Theorems 6.1 and 6.2), while isolated minimizers are stable under perturbations of both the cones and the given functions (see Theorems 6.3 and 6.4). Further, we show, that sense I concepts are stable under perturbations of the objective data (see Theorems 6.1 and 6.3), while sense II concepts are stable under perturbations of both the objective and constrained data (see Theorems 6.2 and 6.4).

The following Theorems 6.1 – 6.4 express the stability properties of the p -minimizers and the isolated minimizers of first order. In their formulation we consider together with the constrained problem (1) also the perturbed problem

$$\min_{\tilde{C}} \tilde{f}(x), \quad \tilde{g}(x) \in -\tilde{K}, \quad (14)$$

where as in (1) we have that $\tilde{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $\tilde{g} : \mathbb{R}^n \rightarrow \mathbb{R}^p$ are $C^{0,1}$ functions and $\tilde{C} \subset \mathbb{R}^m$, $\tilde{K} \subset \mathbb{R}^p$ are closed convex cones.

Theorem 6.1 (Stability of p -minimizers in sense I). *Let x^0 be a p -minimizer in sense I for the constrained problem (1). Then there exists $\delta > 0$, such that for the perturbed problem (14) with $\tilde{C} \subset C(\delta)$, $\tilde{K} = K$, $\tilde{f} = f$, $\tilde{g} = g$, the point x^0 is also a p -minimizer in sense I.*

Proof. By definition the point x^0 is a p -minimizer for the constrained problem (1) if there exists a neighbourhood U of x^0 and a constant $a > 0$ such that if $x \in U \cap g^{-1}(-K)$ then $f(x) - f(x^0) \notin -\text{int } C(a)$. Take δ such that $0 < \delta < a$ and consider the perturbed problem (14) with $\tilde{C} \subset C(\delta)$, $\tilde{K} = K$, $\tilde{f} = f$, $\tilde{g} = g$. From the inclusion $\tilde{C}(a - \delta) \subset C(\delta)(a - \delta) \subset C(a)$ (see Lemma 6.1 below) we see, that $x \in U \cap g^{-1}(-K)$ implies $f(x) - f(x^0) \notin -\text{int } \tilde{C}(a - \delta)$. Therefore x^0 is a p -minimizer in sense I for the perturbed problem. \square

Lemma 6.1. *Let $C \subset \mathbb{R}^m$ be a closed convex cone and $a_1, a_2 > 0$ be two positive numbers. Then $C(a_1)(a_2) \subset C(a_1 + a_2)$.*

Proof. Let $y \in C(a_1)(a_2)$. We must show that $y \in C(a_1 + a_2)$. The case $y \in C(a_1)$ is obvious. Suppose now $y \notin C(a_1)$. Let y' be the orthogonal projection of y on the cone $C(a_1)$ and $d' = D(y, C(a_1)) = \|y - y'\|$. The definition of $C(a_1)(a_2)$ yields $d' \leq a_2\|y\|$ and from the properties of the orthogonal projection we have $\|y'\| \leq \|y\|$. Denote by \bar{y} and \bar{y}' the orthogonal projection respectively of y and y' on C . Put $\bar{d} = D(y, C) = \|y - \bar{y}\|$ and $\bar{d}' = D(y', C) = \|y' - \bar{y}'\|$. Then obviously it holds

$$\begin{aligned} \bar{d} &= \|y - \bar{y}\| \leq \|y - \bar{y}'\| \leq \|y - y'\| + \|y' - \bar{y}'\| \\ &\leq a_2\|y\| + a_1\|y'\| \leq (a_1 + a_2)\|y\|. \end{aligned}$$

Therefore $y \in C(a_1 + a_2)$. □

Theorem 6.2 (Stability of p -minimizers in sense II). *Let x^0 be a p -minimizer in sense II for the constrained problem (1). Then there exists $\delta > 0$, such that if $\tilde{C} \subset C(\delta)$, $\tilde{K} \subset K(x^0)(\delta)$, $\tilde{f} = f$, $\tilde{g} = g$, then the point x^0 is a p -minimizer for the problem $\min_{\tilde{C} \times \tilde{K}} (\tilde{f}(x), \tilde{g}(x))$.*

Proof. We must show that x^0 remains a p -minimizer in sense II under sufficiently small perturbations of C and K . This follows straightforward by applying the proved Theorem 6.1 for the unconstrained problem (13), which gives, that x^0 remains a p -minimizer for (13) perturbed by $\tilde{C} \times \tilde{K} \subset (C \times K(x^0))(\delta)$ and sufficiently small δ . The proof is completed by Lemma 6.2 below. □

Lemma 6.2. *Let $C \subset \mathbb{R}^m$ and $K \subset \mathbb{R}^p$ be closed convex cones. Then for each $\delta > 0$ it holds $C(\delta) \times K(\delta) \subset (C \times K)(\delta)$.*

Proof. Let $y = (y_1, y_2) \in \mathbb{R}^m \times \mathbb{R}^p$. Suppose that $y_1 \in C(\delta)$ and $y_2 \in K(\delta)$. Therefore for the orthogonal projections y'_1 of y_1 on C and y'_2 of y_2 on K it holds $\|y_1 - y'_1\| \leq \delta\|y_1\|$ and $\|y_2 - y'_2\| \leq \delta\|y_2\|$. We get from here

$$D(y, C \times K) \leq (\|y_1 - y'_1\|^2 + \|y_2 - y'_2\|^2)^{1/2} \leq \delta (y_1^2 + y_2^2)^{1/2} = \delta \|y\|.$$

□

Theorem 6.3 (Stability of isolated minimizers in sense I). *Let x^0 be an isolated minimizer of first order in sense I for the constrained problem (1) with f locally Lipschitz. Then there exists $\delta > 0$ and a neighbourhood U of x^0 , such that for the perturbed problem (14) with $\tilde{C} \subset C(\delta)$, $\tilde{K} = K$, $\|\tilde{f}(x) - f(x)\| \leq \delta \|x - x^0\|$ for $x \in U$, $\tilde{g} = g$, the point x^0 is also an isolated minimizer of first order in sense I.*

Proof. Let us make first the remark, that the type of perturbation $\|\tilde{f}(x) - f(x)\| \leq \|x - x^0\|$ gives for $x = x^0$ the equality $\tilde{f}(x^0) = f(x^0)$. The assumption x^0 isolated minimizer of first order means that there exists $A > 0$ and a neighbourhood U of x^0 such that

$$D(f(x) - f(x^0), -C) \geq A \|x - x^0\| \quad \text{for } x \in U \cap g^{-1}(-K). \quad (15)$$

Diminishing eventually U , we may assume that f is Lipschitz with constant L in U . Let $0 < \delta < A/(L + 1)$. If $\tilde{C} \subset C(\delta)$ and $\|\tilde{f}(x) - f(x)\| \leq \|x - x^0\|$ for $x \in U \cap g^{-1}(-K)$ we get

$$\begin{aligned} D(\tilde{f}(x) - \tilde{f}(x^0), -\tilde{C}) &\geq D((f(x) - \tilde{f}(x^0)) + (\tilde{f}(x) - f(x)), -\tilde{C}) \\ &\geq D(f(x) - f(x^0), -\tilde{C}) - \|\tilde{f}(x) - f(x)\| \geq D(f(x) - f(x^0), -C(\delta)) - \delta \|x - x^0\| \\ &\geq D(f(x) - f(x^0), -C) - \delta \|f(x) - f(x^0)\| - \delta \|x - x^0\| \\ &\geq A \|x - x^0\| - \delta L \|x - x^0\| - \delta \|x - x^0\| = (A - \delta(L + 1)) \|x - x^0\|. \end{aligned}$$

This chain of inequalities with the account of $A - \delta(L + 1)$ shows that x^0 is an isolated minimizer of first order for the perturbed problem. We have applied above the Lipschitz property of the oriented distance $D(y^1 + y^2, -C) \geq D(y^1, -C) - \|y^2\|$. We have made also an use of Lemma 6.1 and the given below Lemma 6.3. \square

Lemma 6.3. *Let $C \subset \mathbb{R}^m$ be a closed convex cone and let $y \in \mathbb{R}^m$ and $\delta > 0$ be such that $y \notin C(\delta)$ or $y = 0$. Then $D(y, C) \leq D(y, C(\delta)) + \delta \|y\|$.*

Proof. In the case $y = 0$ the proved inequality becomes obvious, since all terms turn into zero. Suppose now that $y \notin C(\delta)$. Denote by y' the orthogonal projection of y on $C(\delta)$ and by \bar{y} and \bar{y}' the orthogonal projection respectively of y and y' on C . Then

$$\begin{aligned} D(y, C) &= \|y - \bar{y}\| \leq \|y - \bar{y}'\| \leq \|y - y'\| + \|y' - \bar{y}'\| = D(y, C(\delta)) + \|y' - \bar{y}'\| \\ &\leq D(y, C(\delta)) + \delta \|y'\| \leq D(y, C(\delta)) + \delta \|y\|, \end{aligned}$$

which had to be demonstrated. \square

Theorem 6.4 (Stability of isolated minimizers in sense II). *Let x^0 be an isolated minimizer of first order in sense II for the constrained problem (1) with f and g locally Lipschitz. Then there exists $\delta > 0$ and a neighbourhood U of x^0 , such that if $\tilde{C} \subset C(\delta)$, $\tilde{K} \subset K(x^0)(\delta)$, $\|\tilde{f}(x) - f(x)\| \leq \delta \|x - x^0\|$ for $x \in U$, $\|\tilde{g}(x) - g(x)\| \leq \delta \|x - x^0\|$ for $x \in U$, then the point x^0 is an isolated minimizer of first order for the problem $\min_{\tilde{C} \times \tilde{K}} (\tilde{f}(x), \tilde{g}(x))$.*

Proof. We must show, that x^0 remains an isolated minimizer of first order in sense II under sufficiently small perturbations of the problem data. This follows straightforward applying the proved Theorem 6.3 for the unconstrained problem (13). We get that x^0 remains an isolated minimizer perturbing the data so that $\tilde{C} \times \tilde{K} \subset (C \times K(x^0))(\delta)$ and

$$\|(\tilde{f}(x), \tilde{g}(x)) - (f(x), g(x))\| \leq \delta \|x - x^0\|. \quad (16)$$

Now the proof is completed by applying Lemma 6.2 and the proved below Lemma 6.4. \square

Lemma 6.4. *The inequality (16) is satisfied assuming $\|\tilde{f}(x) - f(x)\| \leq \delta_1 \|x - x^0\|$ and $\|\tilde{g}(x) - g(x)\| \leq \delta_1 \|x - x^0\|$ with $\delta_1 = \delta/\sqrt{2}$.*

Proof. This result follows from the chain

$$\begin{aligned} \|(\tilde{f}(x), \tilde{g}(x)) - (f(x), g(x))\| &= \left(\|\tilde{f}(x) - f(x)\|^2 + \|\tilde{g}(x) - g(x)\|^2 \right)^{1/2} \\ &\leq \delta_1 \sqrt{2} \|x - x^0\| = \delta \|x - x^0\|. \end{aligned}$$

\square

We skipped the detailed proof of Theorems 6.2 and 6.4, since each unconstrained problem, in this case problem (13), is a particular case of a constrained problem. Therefore, in principle the proof is a corollary of that of the respective constrained problem. However, there is some peculiarity, which has to be underlined specially. The unconstrained problem (13), that appears in the definition of the sense II concepts, depends on K indirectly through $K(x^0)$. Therefore, it is important to show, that when for the perturbed problem \tilde{K} is close to K , then $\tilde{K}(x^0)$ is also close to $K(x^0)$. Omitting the demonstration, which can be obtained by analysing carefully the structure of the cones $K(x^0)$ and $\tilde{K}(x^0)$, we still would like to underline, that this moment needs some attention.

Comparison Theorems 6.1 and 6.3 a natural questions arises, whether the type of efficiency expressed in Theorem 6.1, namely x^0 p -minimizer in sense I, is preserved under small perturbations admitted in Theorem 6.3, namely small perturbations of C and f . Example 6.2 gives a negative answer of this question, and this is the sense we put in the text “Theorem 6.1 $\not\rightarrow$ Theorem 6.3” in its formulation. Examples 6.3–6.5 give negative answer of similar answers concerning the points having a type of efficiency as in the above theorems. Therefore, Theorems 6.1–Theorem 6.4 describe well the appropriate classes of perturbations preserving the types of efficiency considered in these theorems.

Example 6.2 (Theorem 6.1 $\not\rightarrow$ Theorem 6.3). *Consider the unconstrained problem (2) with $f : \mathbb{R} \rightarrow \mathbb{R}^2$, $f(x) = (x^2, -x^2)$ and $C = \mathbb{R}_+^2$. Then $x_0 = 0$ is a p -minimizer in sense I. Let $\delta > 0$. Consider the perturbed problem $\min_{\tilde{C}} \tilde{f}(x)$ with $\tilde{C} = C$ and $\tilde{f} = f_\delta$,*

where $f_\delta : \mathbb{R} \rightarrow \mathbb{R}^2$, $f_\delta(x) = (x^2 - \delta x, -x^2)$. Then we have $\|f_\delta(x) - f(x)\| = \delta \|x - x_0\|$. At the same time x_0 is not a w -minimizer, and moreover not a p -minimizer in sense I, for the perturbed problem.

In fact, to show that x_0 is not a w -minimizer, it is enough to observe that the function (5) is $\varphi_0(x) = \max(x^2 - \delta x, -x^2) = -x^2$ for $0 \leq x \leq \delta/2$, and obviously does not attain a minimum at x_0 .

Since this example concerns an unconstrained problem, we make the following remark. Each unconstrained problem can be considered as a particular case of a constrained problem. Therefore, the defined sense I and sense II concepts concern also unconstrained problems. As a matter of fact, for an unconstrained problem each p -minimizer is a p -minimizer both in sense I and sense II, and each isolated minimizer of first order is isolated minimizer of first order both in sense I and sense II.

Example 6.3 (Theorem 6.1 \nrightarrow Theorem 6.2). Consider the constrained problem (1) with $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = -x^2$, $C = \mathbb{R}_+$ and $g : \mathbb{R} \rightarrow \mathbb{R}^2$, $g(x) = (x^2, -x)$, $K = \mathbb{R}_+^2$. Then $x_0 = 0$ is a p -minimizer in sense I, since x_0 is the unique feasible point. Let $1 > \delta > 0$. Consider the perturbed problem (14) with $\tilde{C} = C$, $\tilde{f} = f$, $\tilde{g} = g$, and

$$\tilde{K} = K(\delta) = \left\{ (z_1, z_2) \in \mathbb{R}^2 \mid z_2 \geq -\frac{\delta}{\sqrt{1-\delta^2}} z_1, z_2 \geq -\frac{\sqrt{1-\delta^2}}{\delta} z_1 \right\}.$$

Then the set of the feasible points for the perturbed problem is the interval $[0, \delta_1]$, where $\delta_1 = \min\left(\frac{\delta}{\sqrt{1-\delta^2}}, \frac{\sqrt{1-\delta^2}}{\delta}\right)$. Obviously, x_0 is not a w -minimizer, and moreover not a p -minimizer in sense I, for the perturbed problem.

Example 6.4 (Theorem 6.3 \nrightarrow Theorem 6.4). The constrained problem in Example 6.3 has $x_0 = 0$ as an isolated minimizer of first order in sense I. Simultaneously, for the described there perturbed problem the point x_0 is not an isolated minimizer of first order in sense II.

Example 6.5 (Theorem 6.2 \nrightarrow Theorem 6.4). The unconstrained problem in Example 6.2 has $x_0 = 0$ as a p -minimizer in sense II (as we noticed, x_0 can be considered as a p -minimizer both in sense I and sense II). Simultaneously, for the described there perturbed problem x^0 is not a p -minimizer in sense II.

We conclude the section with the following remark. Obviously, it was easier to introduce sense II concepts as solutions of the problem $\min_{C \times K}(f(x), g(x))$. Instead, we have chosen the problem (13) because of the relation of the solutions of this problem with the solutions of the constrained problem (1). This relation has been established by Theorem 4.1. A consequence of this theorem is that under the constrained qualification

$\mathbb{Q}_{0,1}(x^0)$ the feasible point x^0 is an isolated minimizer in sense I for (1) if and only if it is isolated minimizer in sense II. Since sense I p -minimizers and isolated minimizers obey some stability properties (see Theorems 6.1 and 6.3) and the sense I and sense II concepts are interrelated, it is natural to expect that the sense II concepts obey also some stability properties, which has been proved in Theorems 6.2 and 6.4. We can be however a bit dissatisfied from the obtained there results, since our expectations were to obtain a “structural” stability. The “structural” stability we understand in the sense, that as a perturbation of the unconstrained problem (13) it is perhaps more natural to consider not the problem $\min_{\tilde{C} \times \tilde{K}}(\tilde{f}(x), \tilde{g}(x))$ (see Theorems 6.3 and 6.4), but the problem

$$\min_{\tilde{C} \times \tilde{K}(x^0)} (\tilde{f}(x), \tilde{g}(x)). \quad (17)$$

In connection with this remark the following two conjectures can be formulated.

Conjecture 6.1 (Stability of p -minimizers in sense II). *Let x^0 be a p -minimizer in sense II for the constrained problem (1). Then there exists $\delta > 0$, such that for the perturbed problem (14) with $\tilde{C} \subset C(\delta)$, $\tilde{K} \subset K(\delta)$, $\tilde{f} = f$, $\tilde{g} = g$, the point x^0 is also a p -minimizer in sense II, i. e. x^0 is a p -minimizer of first order for problem (17).*

Conjecture 6.2 (Stability of isolated minimizers in sense II). *Let x^0 be an isolated minimizer of first order in sense II for the constrained problem (1) with f and g locally Lipschitz. Then there exists $\delta > 0$ and a neighbourhood U of x^0 , such that for the perturbed problem (14) with $\tilde{C} \subset C(\delta)$, $\tilde{K} \subset K(\delta)$, $\|\tilde{f}(x) - f(x)\| \leq \delta \|x - x^0\|$ for $x \in U$, $\|\tilde{g}(x) - g(x)\| \leq \delta \|x - x^0\|$ for $x \in U$, the point x^0 is also an isolated minimizer of first order, i. e. x^0 is an isolated minimizer of first order for problem (17).*

Unfortunately, Conjectures 6.1 and 6.2 are false, as the following example explains.

Example 6.6. *Consider the constrained problem (1) with $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = 0$, $C = \mathbb{R}_+$, $g : \mathbb{R} \rightarrow \mathbb{R}^2$, $g(x) = (x - 1, |x|)$, $K = \mathbb{R}_+^2$. The point x^0 is the only feasible point, whence x^0 is sense I both p -minimizer and isolated minimizer of first order. It is also sense II both p -minimizer and isolated minimizer of first order, but does not satisfy the conclusions of Conjectures 6.1 and 6.2.*

The associated problem (13) for this example is given by $(f, g) : \mathbb{R} \rightarrow \mathbb{R}^3$, $(f(x), g(x)) = (0, x - 1, |x|)$ and $C \times K(x^0) = \{(y, z_1, z_2) \mid y \geq 0, z_2 \geq 0\}$. It has obviously $x^0 = 0$ as both p -minimizer and isolated minimizer of first order.

The perturbed problem (17) with $\tilde{C} = C$, $\tilde{K} = K(\delta)$, $\tilde{f} = f$, $\tilde{g} = g$, and $\delta > 0$ arbitrary, is given by $(\tilde{f}, \tilde{g}) : \mathbb{R} \rightarrow \mathbb{R}^3$, $(\tilde{f}(x), \tilde{g}(x)) = (0, x - 1, |x|)$ and $\tilde{C} \times \tilde{K}(x^0) = \{(y, z_1, z_2) \mid y \geq 0\}$. For this problem $x^0 = 0$ is neither p -minimizer nor isolated minimizer of first order.

References

- [1] T. Amahroq, A. Taa: *On Lagrange-Kuhn-Tucker multipliers for multiobjective optimization problems*. Optimization 41 (1997), 159–172.
- [2] A. Auslender: *Stability in mathematical programming with nondifferentiable data*. SIAM J. Control Optim. 22 (1984), 239–254.
- [3] A. Balayadi, Y. Sonntag, C. Zălinescu: *Stability of constrained optimization problems*. Nonlinear Anal. 28 No. 8 (1997), 1395–1409.
- [4] M. S. Bazaraa, C. M. Shetty: *Foundation of optimization*. Lecture Notes in Econom. and Math. Systems, Vol. 122, Springer, 1976.
- [5] E. M. Bednarczuk: *A note of lower semicontinuity of minimal points*. Nonlinear Anal. 50 (2002), 285–297.
- [6] H. P. Benson, T. L. Morin: *The vector maximization problem: proper efficiency and stability*. SIAM J. Appl. Math. 32 No. 1 (1977), 64–72.
- [7] J. Borwein: *Proper efficient points for maximization with respect to cones*. SIAM J. Control. Optim. 15 No. 1 (1977), 57–63.
- [8] J. M. Borwein, D. Zhuang: *Superefficiency in vector optimization*. Trans. Amer. Math. Soc. 338 (1993), 105–122.
- [9] M. Ciligot-Travain: *On Lagrange-Kuhn-Tucker multipliers for Pareto optimization problems*. Numer. Funct. Anal. Optim. 15 (1994), 689–693.
- [10] A. M. Geoffrion: *Proper efficiency and the theory of vector maximization*. J. Math. Anal. Appl. 22 No. 3 (1968), 618–630.
- [11] I. Ginchev, A. Guerraggio, M. Rocca: *From scalar to vector optimization*. Appl. Math., to appear.
- [12] I. Ginchev, A. Guerraggio, M. Rocca: *First-order conditions for $C^{0,1}$ constrained vector optimization*. In: F. Giannesi, A. Maugeri (eds.), Variational Analysis and Applications, Proc. Erice, June 20–July 1, 2003, Kluwer, to appear.
- [13] I. Ginchev, A. Hoffmann: *Approximation of set-valued functions by single-valued one*. Discuss. Math. Differ. Incl. Control Optim. 22 (2002), 33–66.
- [14] A. Guerraggio, E. Molho, A. Zaffaroni: *On the notion of proper efficiency in vector optimization*. J. Optim. Theory Appl. 82 (1994), 1–21.

- [15] M. I. Henig: *Proper efficiency with respect to cones*. J. Optim. Theory Appl., 36 (1982), 387–407.
- [16] J.-B. Hiriart-Urruty: *New concepts in nondifferentiable programming*. Analyse non convexe, Bull. Soc. Math. France 60 (1979), 57–85.
- [17] J.-B. Hiriart-Urruty: *Tangent cones, generalized gradients and mathematical programming in Banach spaces*. Math. Oper. Res. 4 (1979), 79–97.
- [18] J. Jahn: *Mathematical vector optimization in partially ordered linear spaces*. Peter Lang, Frankfurt, 1986.
- [19] V. G. Karmanov: *Mathematical programming*. Mir, Moscow, 1989.
- [20] H. W. Kuhn, A. W. Tucker: *Nonlinear Programming*. 2nd Berkeley Symposium on Mathematical Statistics and Probability, pp. 481–492, Berkeley, California, 1951.
- [21] D. T. Luc: *Theory of vector optimization*. Springer, Berlin, 1989.
- [22] O. L. Mangasarian: *Nonlinear programming*. SIAM, Philadelphia 1994.
- [23] E. Miglierina: *Characterization of solutions of multiobjective optimization problems*. Rend. Circ. Mat. Palermo, 50 (2001), 153–164.
- [24] E. Miglierina, E. Molho: *Scalarization and its stability in vector optimization*. J. Optim. Theory Appl., 114 (2002), 657–670.
- [25] V. V. Podinovskiy, V. D. Nogin: *Pareto-optimal solutions of multicriterial problems* (In Russian). Nauka, Moscow, 1982.
- [26] R. T. Rockafellar: *Convex analysis*. Princeton University Press, Princeton, 1970.
- [27] A. Zaffaroni: *Degrees of efficiency and degrees of minimality*. SIAM J. Control Optim. 42 (2003), no. 3, 1071–1086.
- [28] C. Zălinescu: *On two notions of proper efficiency*. In: Brosowski and Martensen (eds.), *Optimization in Mathematical Physics (Oberwolfach 1985)*, pp. 77–86, Peter Lang, Frankfurt, 1987.