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Multivariate hazard orderings of discrete random vectors*

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Abstract

The task of comparing two random vectors with respect to some multivariate stochastic ordering usually involves an infinite number of comparisons. Dyckerhoff and Mosler (1997) proved that, when the random vectors have finite supports, this task, for some orderings, can be simplified by considering only a small finite number of comparisons. In this paper we extend their results to two multivariate hazard rate stochastic orderings.

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1 Introduction

In recent years, multivariate stochastic orderings have become common tools in many fields of application in statistics and probability. Since the seminal work of Lehmann (1955), who first defined the *"usual" multivariate stochastic ordering* $\leq_{\text{st}}$, researchers have given many contributions to the subject, introducing other orderings capable to mathematically describe the tendency of one random vector to assume "larger" values than another, and, especially, studying in detail their properties and many useful applications.

Some interesting examples are represented by the *upper orthant ordering* $\leq_{\text{uo}}$, the *lower orthant ordering* $\leq_{\text{lo}}$, the *upper orthant convex ordering* $\leq_{\text{uocx}}$ and the *lower orthant concave ordering* $\leq_{\text{locv}}$. All these orderings share with the "usual" stochastic ordering a useful characterization in terms of integral inequality with respect to certain classes of functions. An important stochastic ordering which does not have such characterization is the *likelihood ratio ordering* $\leq_{\text{lr}}$.

For a detailed review of the subject the interested reader is referred to Shaked and Shanthikumar (1994) and Müller and Stoyan (2002). A study of integral stochastic orderings in a unified approach may be found in Müller (1997), while a key reference for the properties of the "usual" stochastic ordering is the paper of Kamae et al. (1977). The likelihood ratio ordering was first studied by Karlin and Rinott (1980).

Hu et al. (2003) recently introduced two interesting multivariate stochastic orderings which do not have integral characterization and are strictly related to some multivariate hazard rate notions, so having some useful applications in reliability theory. They are called the *hazard rate ordering* and the *weak hazard rate ordering* and are respectively denoted by $\leq_{\text{whr}}$ and $\leq_{\text{hr}}$. Colangelo et al. (2005a,b) showed their usefulness in the context of positive dependence.

The task of checking any of the orderings considered above for a pair of random vectors usually involves an infinite number of comparisons. Dyckerhoff and Mosler (1997) proved that, when the random vectors have finite supports, this task, for the orderings $\leq_{\text{uo}}$, $\leq_{\text{lo}}$, $\leq_{\text{uocx}}$ and $\leq_{\text{locv}}$, can be efficiently simplified by considering only a relatively small finite number of comparisons. The main aim of this paper is to illustrate that the same result can be proven to hold for $\leq_{\text{whr}}$ as well as for a new stochastic ordering which can be generated with a similar idea and which will be called the *weak multivariate reverse hazard rate ordering* ($\leq_{\text{wrhr}}$).

In particular, in Section 2 the definition of $\leq_{\text{whr}}$ is provided and the new stochastic ordering $\leq_{\text{wrhr}}$ is also briefly introduced and studied. Section 3 contains the main results of the paper, showing that, when comparing two random vectors having finite supports with respect to either $\leq_{\text{whr}}$ or $\leq_{\text{wrhr}}$, a finite number of checks should be made, considering only points in a subset of the grid determined by the combined support of the random vectors.

We use the following conventions throughout the paper. By 'increasing' and 'decreasing' we mean 'non-decreasing' and 'non-increasing,' respectively. For any two n-dimensional vectors $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$, the notation $\boldsymbol{x} \leq \boldsymbol{y}$ means $x_i \leq y_i$, $i = 1, 2, \ldots, n$. The minimum and the maximum operators on $\mathbb{R}$ are respectively denoted by $\wedge$ and $\vee$; furthermore, we use the notation $\boldsymbol{x} \wedge \boldsymbol{y} = (x_1 \wedge y_1, x_2 \wedge y_2, \ldots, x_n \wedge y_n)$, and $\boldsymbol{x} \vee \boldsymbol{y} = (x_1 \vee y_1, x_2 \vee y_2, \ldots, x_n \vee y_n)$. A subset

$L \subseteq \mathbb{R}^n$ is called a sublattice if for any $\boldsymbol{x}, \boldsymbol{y} \in L$ it holds that $\boldsymbol{x} \wedge \boldsymbol{y} \in L$ and $\boldsymbol{x} \vee \boldsymbol{y} \in L$. For every distribution function F of a random vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ we will denote by $\overline{F}$ its corresponding survival function, that is the function defined by $\overline{F}(\boldsymbol{x}) = P(\boldsymbol{X} > \boldsymbol{x}) = 1 - F(\boldsymbol{x})$. Whenever the random vector $\boldsymbol{X}$ has an absolutely continuous distribution function F, its *multivariate hazard function* H is defined by $H(\boldsymbol{x}) = -\log \overline{F}(\boldsymbol{x})$ and the corresponding *hazard gradient* is identified by the vector $\boldsymbol{h}_{\boldsymbol{X}}$ of partial derivatives of H, that is

$$\boldsymbol{h}_{\boldsymbol{X}}(\boldsymbol{x}) = (\boldsymbol{h}_{\boldsymbol{X}}^{(1)}(\boldsymbol{x}), \boldsymbol{h}_{\boldsymbol{X}}^{(2)}(\boldsymbol{x}), \ldots, \boldsymbol{h}_{\boldsymbol{X}}^{(n)}(\boldsymbol{x})) = \left(\frac{\partial}{\partial x_1} H(\boldsymbol{x}), \frac{\partial}{\partial x_2} H(\boldsymbol{x}), \ldots, \frac{\partial}{\partial x_n} H(\boldsymbol{x})\right)$$

for all $\boldsymbol{x} \in \{\boldsymbol{x} : \overline{F}(\boldsymbol{x}) > 0\}$, with the convention that $\boldsymbol{h}_{\boldsymbol{X}}^{(i)}(\boldsymbol{x}) = \infty$ for all $\boldsymbol{x} \in \{\boldsymbol{x} : \overline{F}(\boldsymbol{x}) = 0\}$, for $i = 1, 2, \ldots, n$; see Johnson and Kotz (1975). A simple argument shows that $\boldsymbol{h}_{\boldsymbol{X}}^{(i)}(\boldsymbol{x})$ coincides with the hazard rate of the conditional random variable $[X_i | X_j > x_j, j \neq i]$, $i = 1, 2, \ldots, n$.

2 Multivariate hazard orderings

For two random vectors $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ and $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_n)$, with distribution functions F and G, $\boldsymbol{X}$ is defined to be smaller than $\boldsymbol{Y}$ in the weak multivariate hazard rate ordering (denoted by $\boldsymbol{X} \leq_{\mathrm{whr}} \boldsymbol{Y}$ or $F \leq_{\mathrm{whr}} G$) if

$$\overline{F}(\boldsymbol{y})\overline{G}(\boldsymbol{x}) \leq \overline{F}(\boldsymbol{x})\overline{G}(\boldsymbol{y}) \qquad \text{for all } \boldsymbol{x} \leq \boldsymbol{y}. \tag{1}$$

Whenever F and G are absolutely continuous, condition (1) can be easily proven to be equivalent to the pointwise ordering of the multivariate hazard gradients of $\boldsymbol{X}$ and $\boldsymbol{Y}$ (see Hu et al. (2003), Theorem 2.5), that is $\boldsymbol{X} \leq_{\mathrm{whr}} \boldsymbol{Y}$ if, and only if,

$$\boldsymbol{h}_{\boldsymbol{X}}(\boldsymbol{x}) \geq \boldsymbol{h}_{\boldsymbol{Y}}(\boldsymbol{x}) \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^n. \tag{2}$$

Hu et al. (2003) discuss in detail many preservation properties satisfied by the ordering $\leq_{\mathrm{whr}}$. For example, they prove that it is closed under increasing transformations, marginalization, weak convergence and that an interesting mixing property also holds. Colangelo et al. (2005b) show that $\leq_{\mathrm{whr}}$ can be viewed as a stochastic ordering of positive dependence when the distributions under comparison have the same univariate marginals and, in addition, they verify that it satisfies most of the postulates proposed by Joe (1997) for a positive dependence ordering; they call it the *upper orthant increasing ratio ordering* ($\leq_{\mathrm{uoir}}$).

We will not reproduce here the definitions of the other stochastic orderings mentioned in Section 1 as the reader may find them, together with a detailed discussion of their properties, in Shaked and Shanthikumar (1994) or Müller and Stoyan (2002). Hu et al. (2003) establish that

$$\leq_{\mathrm{lr}} \implies \leq_{\mathrm{whr}} \implies \leq_{\mathrm{uo}} \implies \leq_{\mathrm{uocx}}, \tag{3}$$

while the ordering $\leq_{\mathrm{st}}$ neither implies nor is implied by $\leq_{\mathrm{whr}}$. Example 2.15 in Colangelo et al. (2005b) may be used to show that no relationships hold between $\leq_{\mathrm{whr}}$ and the orderings $\leq_{\mathrm{lo}}$ and $\leq_{\mathrm{locv}}$.

By suitably modifying the stochastic inequality (1) we can now define the following new stochastic ordering; it is strictly related to the *lower orthant decreasing ratio ordering* of positive dependence studied in Colangelo et al. (2005b).

Definition 1. Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ and $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_n)$ be two random vectors with distribution functions F and G, $\boldsymbol{X}$ is defined to be smaller than $\boldsymbol{Y}$ in the weak multivariate reversed hazard rate ordering (denoted by $\boldsymbol{X} \leq_{\text{wrhr}} \boldsymbol{Y}$ or $F \leq_{\text{wrhr}} G$) if

$$F(\boldsymbol{y})G(\boldsymbol{x}) \leq F(\boldsymbol{x})G(\boldsymbol{y}) \qquad \text{for all } \boldsymbol{x} \leq \boldsymbol{y}. \tag{4}$$

It is easy to establish that the ordering $\leq_{\text{wrhr}}$ satisfies the same preservation properties as $\leq_{\text{whr}}$; in addition, the following result gives light to the interesting relationship between the two orderings.

Proposition 2. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two n-dimensional random vectors.*

(i) *If $\boldsymbol{X} \leq_{\text{wrhr}} \boldsymbol{Y}$, then*

$$(\phi_1(X_1), \phi_2(X_2), \ldots, \phi_n(X_n)) \geq_{\text{whr}} (\phi_1(Y_1), \phi_2(Y_2), \ldots, \phi_n(Y_n)) \tag{5}$$

for any sequence of decreasing functions $\phi_1, \phi_2, \ldots, \phi_n : \mathbb{R} \to \mathbb{R}$. Conversely, if condition (5) holds for some strictly decreasing functions $\phi_1, \phi_2, \ldots, \phi_n : \mathbb{R} \to \mathbb{R}$, then $\boldsymbol{X} \leq_{\text{wrhr}} \boldsymbol{Y}$.

(ii) *If $\boldsymbol{X} \leq_{\text{whr}} \boldsymbol{Y}$, then*

$$(\phi_1(X_1), \phi_2(X_2), \ldots, \phi_n(X_n)) \geq_{\text{wrhr}} (\phi_1(Y_1), \phi_2(Y_2), \ldots, \phi_n(Y_n)) \tag{6}$$

for any sequence of decreasing functions $\phi_1, \phi_2, \ldots, \phi_n : \mathbb{R} \to \mathbb{R}$. Conversely, if condition (6) holds for some strictly decreasing functions $\phi_1, \phi_2, \ldots, \phi_n : \mathbb{R} \to \mathbb{R}$, then $\boldsymbol{X} \leq_{\text{whr}} \boldsymbol{Y}$.

In analogy with the univariate case, for a random vector $\boldsymbol{X}$ with absolutely continuous distribution function F, we define its *multivariate reversed hazard function* R as $R(\boldsymbol{x}) = -\log F(\boldsymbol{x})$ and the corresponding *reversed hazard gradient* as the vector $\boldsymbol{r_X}$ of partial derivatives of R, that is

$$\boldsymbol{r_X}(\boldsymbol{x}) = (\boldsymbol{r}^{(1)}_{\boldsymbol{X}}(\boldsymbol{x}), \boldsymbol{r}^{(2)}_{\boldsymbol{X}}(\boldsymbol{x}), \ldots, \boldsymbol{r}^{(n)}_{\boldsymbol{X}}(\boldsymbol{x})) = \left(\frac{\partial}{\partial x_1}R(\boldsymbol{x}), \frac{\partial}{\partial x_2}R(\boldsymbol{x}), \ldots, \frac{\partial}{\partial x_n}R(\boldsymbol{x})\right)$$

for all $\boldsymbol{x} \in \{\boldsymbol{x} : F(\boldsymbol{x}) > 0\}$, with the convention that $\boldsymbol{r}^{(i)}_{\boldsymbol{X}}(\boldsymbol{x}) = \infty$ for all $\boldsymbol{x} \in \{\boldsymbol{x} : F(\boldsymbol{x}) = 0\}$, for $i = 1, 2, \ldots, n$. It is not difficult to see that $\boldsymbol{r}^{(i)}_{\boldsymbol{X}}(\boldsymbol{x})$ coincides with the reversed hazard rate of the conditional random variable $[X_i | X_j \leq x_j, j \neq i]$, $i = 1, 2, \ldots, n$. The following result, which is the analogous of the characterization of $\leq_{\text{whr}}$ stated in (2), provides some theoretical justification to the name given to the stochastic ordering specified by condition (4).

Proposition 3. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two n-dimensional random vectors with reversed hazard gradients $\boldsymbol{r_X}$ and $\boldsymbol{r_Y}$. Then $\boldsymbol{X} \leq_{\text{wrhr}} \boldsymbol{Y}$ if, and only if,*

$$\boldsymbol{r_X}(\boldsymbol{x}) \leq \boldsymbol{r_Y}(\boldsymbol{x}) \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^n.$$

We close the section underlying that (3), together with Proposition 2 and well known properties of the other orderings considered above, may be used to establish that

$$\leq_{\mathrm{lr}} \implies \leq_{\mathrm{wrhr}} \implies \leq_{\mathrm{lo}} \implies \leq_{\mathrm{locv}},$$

while no relationship can hold between $\leq_{\mathrm{wrhr}}$ and either one of $\leq_{\mathrm{st}}$, $\leq_{\mathrm{uo}}$ and $\leq_{\mathrm{uocx}}$.

3 Comparing discrete random vectors

The task of checking the orderings $\leq_{\mathrm{whr}}$ and $\leq_{\mathrm{wrhr}}$ for a pair of random vectors usually involves an infinite number of comparisons, as conditions (1) and (4) need to be verified for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ such that $\boldsymbol{x} \leq \boldsymbol{y}$. In this section we show that, when the supports of the random vectors under comparison are finite, this task can be efficiently simplified by considering only a relatively small finite number of comparisons. Similar results were obtained by Dyckerhoff and Mosler (1997) for the stochastic orderings $\leq_{\mathrm{uo}}$, $\leq_{\mathrm{lo}}$, $\leq_{\mathrm{uocx}}$ and $\leq_{\mathrm{locv}}$.

Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two random vectors having probability distributions with finite support. Their combined support is defined as the set

$$S_{\boldsymbol{X},\boldsymbol{Y}} = \{\boldsymbol{x} \in \mathbb{R}^n : P(\boldsymbol{X} = \boldsymbol{x}) > 0\} \cup \{\boldsymbol{x} \in \mathbb{R}^n : P(\boldsymbol{Y} = \boldsymbol{x}) > 0\}.$$

The following example shows that verifying conditions (1) and (4) for all $\boldsymbol{x}, \boldsymbol{y} \in S_{\boldsymbol{X},\boldsymbol{Y}}$ is not sufficient when checking for the orderings $\leq_{\mathrm{whr}}$ and $\leq_{\mathrm{wrhr}}$.

Example 4. Let (X_1, X_2) and (Y_1, Y_2) be random vectors with probability mass functions

$x_1 \backslash x_2$	0	1	2
2	0	1/3	0
1	0	0	1/3
0	1/3	0	0

and

$y_1 \backslash y_2$	0	1	2
2	0	0	1/3
1	1/3	0	0
0	0	1/3	0

Denoting by F and G the distribution functions of $\boldsymbol{X}$ and $\boldsymbol{Y}$, a simple calculation shows that (1) holds on the combined support $S_{\boldsymbol{X},\boldsymbol{Y}} = \{(0,0), (0,1), (1,0), (1,2), (2,1), (2,2)\}$, while it fails when letting, for example, $\boldsymbol{x} = (-1,-1)$ and $\boldsymbol{y} = (0,0)$, so that $\boldsymbol{X} \not\leq_{\mathrm{whr}} \boldsymbol{Y}$. By Proposition 2 it also follows that $-\boldsymbol{X} \not\geq_{\mathrm{wrhr}} -\boldsymbol{Y}$, while it is trivial to verify that (4) is satisfied on the set $S_{-\boldsymbol{X},-\boldsymbol{Y}}$. ◀

We will now prove that, although considering the combined support of the two random vectors is not sufficient, it is always possible to restrict our attention to a certain subset of the grid of points determined by the combined supports of the single components of the vectors. Before stating the result, some further notation is introduced.

For any $\boldsymbol{s} \in S_{\boldsymbol{X},\boldsymbol{Y}}$ define $p_{\boldsymbol{X}}(\boldsymbol{s}) = P(\boldsymbol{X} = \boldsymbol{s})$ and, analogously, $p_{\boldsymbol{Y}}(\boldsymbol{s}) = P(\boldsymbol{Y} = \boldsymbol{s})$. For all $\boldsymbol{a} \in \mathbb{R}^n$, let $L(\boldsymbol{a}) = \{\boldsymbol{s} \in S_{\boldsymbol{X},\boldsymbol{Y}} : \boldsymbol{s} \leq \boldsymbol{a}\}$ and $U(\boldsymbol{a}) = \{\boldsymbol{s} \in S_{\boldsymbol{X},\boldsymbol{Y}} : \boldsymbol{s} \geq \boldsymbol{a}\}$, so that

$$P(\boldsymbol{X} \leq \boldsymbol{a}) = \sum_{\boldsymbol{s} \in L(\boldsymbol{a})} p_{\boldsymbol{X}}(\boldsymbol{s}) \equiv \Delta^{\mathrm{lo}}_{\boldsymbol{X}}(\boldsymbol{a}) \quad \text{and} \quad P(\boldsymbol{Y} \leq \boldsymbol{a}) = \sum_{\boldsymbol{s} \in L(\boldsymbol{a})} p_{\boldsymbol{Y}}(\boldsymbol{s}) \equiv \Delta^{\mathrm{lo}}_{\boldsymbol{Y}}(\boldsymbol{a}),$$

$$P(\boldsymbol{X} \geq \boldsymbol{a}) = \sum_{\boldsymbol{s} \in U(\boldsymbol{a})} p_{\boldsymbol{X}}(\boldsymbol{s}) \equiv \Delta^{\mathrm{uo}}_{\boldsymbol{X}}(\boldsymbol{a}) \quad \text{and} \quad P(\boldsymbol{Y} \geq \boldsymbol{a}) = \sum_{\boldsymbol{s} \in U(\boldsymbol{a})} p_{\boldsymbol{Y}}(\boldsymbol{s}) \equiv \Delta^{\mathrm{uo}}_{\boldsymbol{Y}}(\boldsymbol{a}).$$

Therefore, it is easy to see, in the new notation, that

(i) $\boldsymbol{X} \leq_{\mathrm{whr}} \boldsymbol{Y}$ if, and only if,

$$\Delta^{\mathrm{uo}}_{\boldsymbol{X}}(\boldsymbol{b})\Delta^{\mathrm{uo}}_{\boldsymbol{Y}}(\boldsymbol{a}) \leq \Delta^{\mathrm{uo}}_{\boldsymbol{X}}(\boldsymbol{a})\Delta^{\mathrm{uo}}_{\boldsymbol{Y}}(\boldsymbol{b}) \qquad \text{for all } \boldsymbol{a} \leq \boldsymbol{b}; \tag{7}$$

(ii) $\boldsymbol{X} \leq_{\mathrm{wrhr}} \boldsymbol{Y}$ if, and only if,

$$\Delta^{\mathrm{lo}}_{\boldsymbol{X}}(\boldsymbol{b})\Delta^{\mathrm{lo}}_{\boldsymbol{Y}}(\boldsymbol{a}) \leq \Delta^{\mathrm{lo}}_{\boldsymbol{X}}(\boldsymbol{a})\Delta^{\mathrm{lo}}_{\boldsymbol{Y}}(\boldsymbol{b}) \qquad \text{for all } \boldsymbol{a} \leq \boldsymbol{b}. \tag{8}$$

Let now $\vee$ and $\wedge$ be the lattice operators as defined in Section 1, and, for any given nonempty set $T = \{\boldsymbol{t}^1, \boldsymbol{t}^2, \ldots, \boldsymbol{t}^k\} \subset \mathbb{R}^n$, recall that the *meet* and the *join* of T are respectively defined as the points

$$\begin{aligned} \bigwedge_{l=1,2,\ldots,k} \boldsymbol{t}^l &= (t_1^1 \wedge t_1^2 \wedge \cdots \wedge t_1^k, t_2^1 \wedge t_2^2 \wedge \cdots \wedge t_2^k, \ldots, t_n^1 \wedge t_n^2 \wedge \cdots \wedge t_n^k) \\ &= \big(\min_{l=1,2,\ldots,k} t_1^l, \min_{l=1,2,\ldots,k} t_2^l, \ldots, \min_{l=1,2,\ldots,k} t_n^l\big), \end{aligned}$$

and

$$\begin{aligned} \bigvee_{l=1,2,\ldots,k} \boldsymbol{t}^l &= (t_1^1 \vee t_1^2 \vee \cdots \vee t_1^k, t_2^1 \vee t_2^2 \vee \cdots \vee t_2^k, \ldots, t_n^1 \vee t_n^2 \vee \cdots \vee t_n^k) \\ &= \big(\max_{l=1,2,\ldots,k} t_1^l, \max_{l=1,2,\ldots,k} t_2^l, \ldots, \max_{l=1,2,\ldots,k} t_n^l\big); \end{aligned}$$

see Birkhoff (1940). The *meet semi-lattice* $M(T)$ of T is the smallest set of $\mathbb{R}^n$ containing T which is closed under the *meet* operator and it is equivalent to the collection of the *meets* of all subsets of T. Similarly, the *join semi-lattice* $J(T)$ of T is the smallest set of $\mathbb{R}^n$ containing T which is closed under the *join* operator and it is equivalent to the collection of the *joins* of all subsets of T. For a proof of the previous statements, we refer again to Birkhoff (1940). Dyckerhoff and Mosler (1997) discuss some simple examples clarifying the construction of meet and join semi-lattices.

We will now show that, in our setting, when checking for the orderings $\leq_{\mathrm{uoir}}$ and $\leq_{\mathrm{lodr}}$ it suffices to verify conditions (7) and (8) for all points respectively lying in $M(S_{\boldsymbol{X},\boldsymbol{Y}})$ and $J(S_{\boldsymbol{X},\boldsymbol{Y}})$.

Theorem 5. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two random vectors with finite supports. Then*

(i) $\boldsymbol{X} \leq_{\mathrm{whr}} \boldsymbol{Y}$ *if, and only if,* (7) *holds for all $\boldsymbol{a}, \boldsymbol{b} \in M(S_{\boldsymbol{X},\boldsymbol{Y}})$ such that $\boldsymbol{a} \leq \boldsymbol{b}$;*

(ii) $\boldsymbol{X} \leq_{\mathrm{wrhr}} \boldsymbol{Y}$ *if, and only if,* (8) *holds for all $\boldsymbol{a}, \boldsymbol{b} \in J(S_{\boldsymbol{X},\boldsymbol{Y}})$ such that $\boldsymbol{a} \leq \boldsymbol{b}$.*

Proof. We will first consider statement (i); clearly, only sufficiency needs to be established. Fix $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$ with $\boldsymbol{a} \leq \boldsymbol{b}$, then $U(\boldsymbol{a}) \supseteq U(\boldsymbol{b})$; in addition, (7) holds trivially when $U(\boldsymbol{b}) = \emptyset$, so that we can always assume $U(\boldsymbol{b}) \neq \emptyset$. Define

$$\underline{\boldsymbol{a}} = \bigwedge_{\boldsymbol{s} \in U(\boldsymbol{a})} \boldsymbol{s} \quad \text{and} \quad \underline{\boldsymbol{b}} = \bigwedge_{\boldsymbol{s} \in U(\boldsymbol{b})} \boldsymbol{s}; \tag{9}$$

we now show that $U(\underline{\boldsymbol{a}}) = U(\boldsymbol{a})$ and $U(\underline{\boldsymbol{b}}) = U(\boldsymbol{b})$. In fact, if $\boldsymbol{t} \in U(\boldsymbol{a})$ then $\boldsymbol{t} \geq \bigwedge_{\{\boldsymbol{s} \in S_{\boldsymbol{X},\boldsymbol{Y}} : \boldsymbol{s} \geq \boldsymbol{a}\}} s = \underline{\boldsymbol{a}}$ and, conversely, if $\boldsymbol{t} \in U(\underline{\boldsymbol{a}})$ then $\boldsymbol{t} \geq \underline{\boldsymbol{a}} = \bigwedge_{\{\boldsymbol{s} \in S_{\boldsymbol{X},\boldsymbol{Y}} : \boldsymbol{s} \geq \boldsymbol{a}\}} \geq \boldsymbol{a}$. Hence $\Delta^{\text{uo}}_{\boldsymbol{X}}(\boldsymbol{a}) = \sum_{\boldsymbol{s} \in U(\boldsymbol{a})} p_{\boldsymbol{X}}(\boldsymbol{s}) = \sum_{\boldsymbol{s} \in U(\underline{\boldsymbol{a}})} p_{\boldsymbol{X}}(\boldsymbol{s}) = \Delta^{\text{uo}}_{\boldsymbol{X}}(\underline{\boldsymbol{a}})$ and, similarly, $\Delta^{\text{uo}}_{\boldsymbol{X}}(\boldsymbol{b}) = \Delta^{\text{uo}}_{\boldsymbol{X}}(\underline{\boldsymbol{b}})$, $\Delta^{\text{uo}}_{\boldsymbol{Y}}(\boldsymbol{a}) = \Delta^{\text{uo}}_{\boldsymbol{Y}}(\underline{\boldsymbol{a}})$ and $\Delta^{\text{uo}}_{\boldsymbol{Y}}(\boldsymbol{b}) = \Delta^{\text{uo}}_{\boldsymbol{X}}(\underline{\boldsymbol{b}})$. Therefore, it suffices to check condition (7) for all points $\underline{\boldsymbol{a}}, \underline{\boldsymbol{b}}$ of type (9) such that $\underline{\boldsymbol{a}} \leq \underline{\boldsymbol{b}}$.

It remains to prove that the class of such $\underline{\boldsymbol{a}}, \underline{\boldsymbol{b}}$ coincides with the set

$$\{\boldsymbol{a}, \boldsymbol{b} \in M(S_{\boldsymbol{X},\boldsymbol{Y}}) : \boldsymbol{a} \leq \boldsymbol{b}\}.$$

To see this, let $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$ be defined as in (9) for some $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$ with $U(\boldsymbol{b}) \neq \emptyset$; notice that if $U(\boldsymbol{b}) \neq \emptyset$ then $U(\boldsymbol{a}) \neq \emptyset$, so that they must both belong to $M(S_{\boldsymbol{X},\boldsymbol{Y}})$ since $U(\boldsymbol{a})$ and $U(\boldsymbol{b})$ are nonempty subsets of $S_{\boldsymbol{X},\boldsymbol{Y}}$. In addition, it clearly holds that $\underline{\boldsymbol{a}} \leq \underline{\boldsymbol{b}}$.

Conversely, let $\boldsymbol{a} = \bigwedge_D s$ and $\boldsymbol{b} = \bigwedge_G s$ with $\boldsymbol{a} \leq \boldsymbol{b}$, where D and G are nonempty subsets of $S_{\boldsymbol{X},\boldsymbol{Y}}$. Then there always exist two other subsets D' and G' of $S_{\boldsymbol{X},\boldsymbol{Y}}$ of the form $U(\boldsymbol{a'})$ and $U(\boldsymbol{b'})$ for some $a', b' \in \mathbb{R}^n$ such that $D' \supseteq D$, $G' \supseteq G$, $\boldsymbol{a} = \bigwedge_{D'} s$ and $\boldsymbol{b} = \bigwedge_{G'} s$. Therefore $\boldsymbol{a}$ and $\boldsymbol{b}$ are of type (9).

Let us now consider statement (ii). For any set $A \subseteq \mathbb{R}^n$, denote $-A = \{\boldsymbol{x} \in \mathbb{R}^n : -\boldsymbol{x} \in A\}$. Recall that, by Proposition 2, $\boldsymbol{X} \leq_{\text{wrhr}} \boldsymbol{Y}$ if, and only if, $-\boldsymbol{X} \geq_{\text{whr}} -\boldsymbol{Y}$ and notice that the set $-S_{\boldsymbol{X},\boldsymbol{Y}}$ represents the combined support of $-\boldsymbol{X}$ and $-\boldsymbol{Y}$; in addition it clearly holds $M(-S_{\boldsymbol{X},\boldsymbol{Y}}) = -J(S_{\boldsymbol{X},\boldsymbol{Y}})$ since $(-x) \wedge (-y) = -(x \vee y)$ for all $x, y \in \mathbb{R}$.

Hence $\boldsymbol{X} \leq_{\text{wrhr}} \boldsymbol{Y}$ if, and only if,

$$\Delta^{\text{uo}}_{-\boldsymbol{X}}(\boldsymbol{b})\Delta^{\text{uo}}_{-\boldsymbol{Y}}(\boldsymbol{a}) \geq \Delta^{\text{uo}}_{-\boldsymbol{X}}(\boldsymbol{a})\Delta^{\text{uo}}_{-\boldsymbol{Y}}(\boldsymbol{b})$$

for all $\boldsymbol{a}, \boldsymbol{b} \in -J(S_{\boldsymbol{X},\boldsymbol{Y}})$ such that $\boldsymbol{a} \leq \boldsymbol{b}$, which is clearly equivalent to

$$\Delta^{\text{lo}}_{\boldsymbol{X}}(-\boldsymbol{b})\Delta^{\text{lo}}_{\boldsymbol{Y}}(-\boldsymbol{a}) \geq \Delta^{\text{lo}}_{\boldsymbol{X}}(-\boldsymbol{a})\Delta^{\text{lo}}_{\boldsymbol{Y}}(-\boldsymbol{b})$$

for all $\boldsymbol{a}, \boldsymbol{b} \in -J(S_{\boldsymbol{X},\boldsymbol{Y}})$ such that $\boldsymbol{a} \leq \boldsymbol{b}$. Noticing that $-\boldsymbol{a}, -\boldsymbol{b} \in J(S_{\boldsymbol{X},\boldsymbol{Y}})$ and $-\boldsymbol{b} \leq -\boldsymbol{a}$ completes the proof. □

An alternative proof of Theorem 5 can be given using results in Dyckerhoff and Mosler (1997, Theorems 1 and 3) and Hu et al. (2003, Theorem 2.4), but the argument would be lengthy and less elegant.

We now briefly remark on the size of the sets $M(T)$ and $J(T)$ for any finite set $T \subset \mathbb{R}^n$. In fact Dyckerhoff and Mosler (1997) notice that, since each meet and each join is the combination of at most n points in $\mathbb{R}^n$, $M(T)$ and $J(T)$ respectively consist of all meets and joins of at most n elements of T. Therefore, for two discrete random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$, the size of $M(S_{\boldsymbol{X},\boldsymbol{Y}})$ and $J(S_{\boldsymbol{X},\boldsymbol{Y}})$ is not larger than

$N = \sum_{i=1}^{n} \binom{k}{i}$, where k is the number of elements in their combined support $S_{\boldsymbol{X},\boldsymbol{Y}}$. Correspondingly, the total number of comparisons for $\leq_{\text{whr}}$ and $\leq_{\text{wrhr}}$ cannot exceed $\binom{N}{2}$, and when k is large with respect to n, this entails a considerable computational gain compared to considering the grid determined by the combined supports of the single components of the random vectors. We now discuss a simple example giving light to the previous statements.

Example 6. Let (X_1, X_2) and (Y_1, Y_2) be random vectors with probability mass functions

$$
\begin{array}{c|ccc}
2 & 0 & 0 & 1/4 \\
1 & 1/4 & 1/4 & 0 \\
0 & 0 & 1/4 & 0 \\
\hline
x_1 \diagup x_2 & 0 & 1 & 2
\end{array}
\qquad \text{and} \qquad
\begin{array}{c|cc}
3 & 0 & 3/4 \\
1 & 1/4 & 0 \\
\hline
y_1 \diagup y_2 & 1 & 3
\end{array}
$$

Clearly $S_{\boldsymbol{X},\boldsymbol{Y}} = \{(0,1), (1,0), (1,1), (2,2), (3,3)\}$ and $M(S_{\boldsymbol{X},\boldsymbol{Y}}) = S_{\boldsymbol{X},\boldsymbol{Y}} \cup \{(0,0)\}$, whose cardinality is much smaller than the grid $\{0,1,2,3\} \times \{0,1,2,3\}$. A simple calculation then shows that $\boldsymbol{X} \leq_{\text{whr}} \boldsymbol{Y}$. ◀

It should be also stressed that the results continue to hold when considering $\leq_{\text{whr}}$ and $\leq_{\text{wrhr}}$ as defining positive dependence stochastic orderings (see Colangelo et al. (2005b)) by letting the random vectors under comparison have common univariate marginal distributions. In particular, Example 4 and Theorem 5 can be adapted to show that also in this context verifying conditions (1) and (4) on $M(S_{\boldsymbol{X},\boldsymbol{Y}})$ and $J(S_{\boldsymbol{X},\boldsymbol{Y}})$ is sufficient, while on $S_{\boldsymbol{X},\boldsymbol{Y}}$ it is not. Clearly, in this situation the saving in the number of comparisons with respect to considering the whole grid would be less beneficial, as fixing the marginals entails a strong restriction on the form of the combined support.

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