

Giovanni Fonseca

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2005/3



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Printed in Italy in June 2005
Università degli Studi dell'Insubria
Via Ravasi 2, 21100 Varese, Italy

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On the stability of nonlinear ARMA models

Giovanni Fonseca
Dipartimento di Economia
Università degli Studi dell'Insubria - Varese
via Ravasi, 2
21100 Varese
E-mail: gfonseca@eco.uninsubria.it

Abstract

In the present paper we study the stability of a class of nonlinear ARMA models. We derive a sufficient condition to ensure the geometric ergodicity and we apply it to a very general threshold ARMA model imposing a mild assumption on the thresholds.

Keywords: Nonlinear ARMA models, threshold ARMA processes, stationary processes, geometric ergodicity.

AMS 2000 SCI: 62M10, 60G10, 60J05.

1 Introduction

In the present paper we consider the class of nonlinear ARMA models

$$X_n = f(\underline{X}_{n-1}) + \sum_{j=1}^m \left(\sum_{i=1}^q b_i^{(j)} e_{n-i} \right) \mathbb{1}_{R_j}(\underline{X}_{n-1}) + e_n \quad (1)$$

where $\underline{X}_{n-1} = (X_{n-1}, \dots, X_{n-p})'$, $f : \mathbb{R}^p \rightarrow \mathbb{R}$ is a locally bounded function, $b_i^{(j)} \in \mathbb{R}$, for all i and j , the sets R_j , for $j = 1, \dots, m$, form a partition of the space \mathbb{R}^p , \underline{X}_0 is a suitable random variable on \mathbb{R}^p and $\mathbb{1}_B(\cdot)$ is the indicator function of the set B .

In particular we apply our results to a time series model with different ARMA regimes, depending on the past history of the series itself, which, in the literature, is usually called Self-Exciting Threshold ARMA (SETARMA) model and defined by

$$X_n = \sum_{j=1}^m \left\{ \sum_{i=1}^p a_i^{(j)} X_{n-i} + \sum_{i=1}^q b_i^{(j)} e_{n-i} \right\} \mathbb{1}_{R_j}(\underline{X}_{n-1}) + e_n \quad (2)$$

where $a_i^{(j)}, b_i^{(j)} \in \mathbb{R}$, for all i and j and the sets R_j , for $j = 1, \dots, m$, as above. Furthermore we suppose that every ARMA regime involved in the model is in reduced form. Note that (2) is a particular case of (1) with

$$f(\underline{X}_{n-1}) = \sum_{j=1}^m \left(\sum_{i=1}^p a_i^{(j)} X_{n-i} \right) \mathbb{1}_{R_j}(\underline{X}_{n-1}).$$

From now on, we assume that

- (E) $\{e_n\}_{n \in \mathbb{N}}$ is a sequence of independent and identically distributed (i.i.d.) random variables, independent from \underline{X}_0 . They are absolutely continuous with respect to the Lebesgue measure μ on \mathbb{R} with positive density function $p(\cdot)$ on \mathbb{R} and $\mathbb{E}[|e_1|] = M < +\infty$.

Various results concerning threshold models have been obtained since the publication of the seminal work on the subject by Tong (1983). Brockwell, Liu and Tweedie (1992) consider the particular case of (2) with $b_i^{(j)} = b_i$, for all i and j , a continuity assumption on the conditional mean $\mathbb{E}[X_n | \underline{X}_{n-1} = \underline{x}]$ and $R_j = \mathbb{R} \times \dots \times r_j \times \dots \times \mathbb{R} \in \mathbb{R}^p$, where the r_j 's form a partition of \mathbb{R} . For such a model they obtain a sufficient condition for the existence of a unique strictly stationary measure without using Markov chain arguments. Indeed the process $\{X_n\}$ in (2) is not a Markov chain but, as in other cases like linear and nonlinear autoregressive processes, we can build a suitable Markov chain, strictly related to the original model, so to be able to transfer the results from one process to the other. Then, one of the difficulties in obtaining stability results for this model lies on the fact that it is not straightforward to find out if the related Markov process is irreducible or continuous, and these are two fundamental properties in applying the 'drift'-criteria for ergodicity and geometric ergodicity. Therefore the study of the stability of nonlinear ARMA time series models is less easier than that of nonlinear autoregressive models because of the more complicate probabilistic structure of the related Markovian representation of the process as shown in Section 2 (a full account on the subject of 'drift'-conditions is in Meyn and Tweedie (1993)). For these reasons other works focus on conditions for the existence of a stationary measure instead of on the stronger result of ergodicity. Nonetheless Ling (1999) and Lee (2000) impose strong assumptions on the process, like, respectively, a continuity condition on the boundary of the thresholds and the weak Feller hypothesis. Even in Cline and Pu (1999)

irreducibility and T-continuity are two basic assumptions in obtaining their stability results for threshold-like ARMA models.

In the present paper we provide sufficient conditions for the irreducibility and the T-continuity of the Markovian representation of the fairly general class of nonlinear ARMA models (1). Hence, under the same conditions, we prove the geometric ergodicity of the process. Furthermore, we apply these results to the general threshold ARMA model (2) imposing only the mild assumption that the origin belongs to the interior of one of the partition sets R_j .

The paper is organized as follows. In Section 2 we define a vectorial Markovian representation of (1) and (2). In Section 3 we state sufficient conditions for the irreducibility and T-continuity of the Markovian representations. In Section 4 we prove the geometric ergodicity of the processes under investigation.

2 Markovian representation

If we define the vector $Y_n = (\underline{X}'_n, \underline{e}'_n)'$, where $\underline{e}_n = (e_n, \dots, e_{n-q+1})'$, we can rewrite model (1) in the following Markovian vectorial representation

$$Y_n = F(Y_{n-1}) + \sum_{j=1}^m A^{(j)} Y_{n-1} \mathbb{1}_{R_j^*}(Y_{n-1}) + \underline{c} e_n \quad (3)$$

where $F(Y_{n-1}) = (f(\underline{X}_{n-1}), 0, \dots, 0)' \in \mathbb{R}^{p+q}$, $\underline{c} = (1, \dots, 0, 1, 0, \dots, 0)' \in \mathbb{R}^{p+q}$, with the 2^{nd} 1 in the $(p+1)^{th}$ position, $R_j^* = (R_j \times \mathbb{R}^q) \subset \mathbb{R}^{p+q}$ and

$$A^{(j)} = \begin{bmatrix} \underline{0}'_{p-1} & 0 & b_1^{(j)}, \dots, b_{q-1}^{(j)} & b_q^{(j)} \\ I_{(p-1) \times (p-1)} & \underline{0}_{p-1} & 0_{(p-1) \times (q-1)} & \underline{0}_{p-1} \\ \underline{0}'_{p-1} & 0 & \underline{0}'_{q-1} & 0 \\ 0_{(q-1) \times (p-1)} & \underline{0}_{q-1} & \underline{I}_{(q-1) \times (q-1)} & \underline{0}_{q-1} \end{bmatrix}.$$

Note that the $(p+1)^{th}$ row of each matrix $A^{(j)}$ is a null row because it corresponds to e_n in the vector Y_n , and that $Y_{n-1} \in R_j^* = (R_j \times \mathbb{R}^q)$ if and only if $\underline{X}_{n-1} \in R_j$, since $\underline{e}_{n-1} \in \mathbb{R}^q$ by definition.

We obtain the Markovian representation of (2) from (3) with $F(Y_{n-1}) = \underline{0}'_{p+q}$ and

$$A^{(j)} = \begin{bmatrix} a_1^{(j)}, \dots, a_{p-1}^{(j)} & a_p^{(j)} & b_1^{(j)}, \dots, b_{q-1}^{(j)} & b_q^{(j)} \\ I_{(p-1) \times (p-1)} & \underline{0}_{p-1} & 0_{(p-1) \times (q-1)} & \underline{0}_{p-1} \\ \underline{0}'_{p-1} & 0 & \underline{0}'_{q-1} & 0 \\ 0_{(q-1) \times (p-1)} & \underline{0}_{q-1} & \underline{I}_{(q-1) \times (q-1)} & \underline{0}_{q-1} \end{bmatrix}.$$

Defining the operator

$$Dy = F(y) + \sum_{j=1}^m A^{(j)} \mathbb{1}_{R_j^*}(y) y,$$

we may rewrite (3) as follows

$$Y_n = DY_{n-1} + \underline{c} e_n.$$

Now let $B \in \mathcal{B}(\mathbb{R}^{p+q})$ and $y = (y_1, \dots, y_{p+q})' \in \mathbb{R}^{p+q}$ be respectively a set and a point on the state space of the Markov chain $\{Y_n\}$ and define the section

$$B_y = \{w \in \mathbb{R}^{p+q} : w \in B, w_i = y_i, i \neq 1, p+1\},$$

that is, a subset of B with $p + q - 2$ coordinates fixed and only the 1st and the $(p + 1)^{th}$ coordinates allowed to vary accordingly to the definition of the set B . Let $v_{1,p+1} : \mathbb{R}^{p+q} \rightarrow \mathbb{R}^2$ be the projection map onto the 1st and $(p + 1)^{th}$ coordinates and $v_{1,p+1}(B_y) = B_y^2$. It is clear that if the past Y_{n-1} is given and fixed, the probabilistic behavior of Y_n is completely dependent on that of e_n . Indeed, we have that

$$P(y, B) = \Pr \left[(e_n, e_n)' \in B_{Dy}^2 - v_{1,p+1}(Dy) \right].$$

It is interesting to point out that the 2nd coordinate of $v_{1,p+1}(Dy)$ (that is, the $(p+1)^{th}$ coordinate of Dy) is 0. In terms of the density of e_n we have therefore

$$P(y, B) = \int_Q p(u) du \tag{4}$$

where the set $Q \in \mathbb{R}^1$ is defined by

$$Q = \{w \in \mathbb{R}^1 : (w, w)' \in B_{Dy}^2 - [v_1(Dy), 0]'\},$$

with $v_1 : \mathbb{R}^{p+q} \rightarrow \mathbb{R}^1$ the projection map onto the 1st coordinate.

The sequence $\{Y_n\}$, together with the transition probabilities (4), forms a Markov chain. Moreover, by assumption (E) the subchain $\{Y_n\}$ is obviously aperiodic.

3 Irreducibility and T-continuity of the model

The Markov chain $\{Y_n\}$ is φ -irreducible, with φ a σ -finite measure on $\mathcal{B}(\mathbb{R}^{p+q})$, if

$$\sum_{i=1}^{\infty} P^i(y, B) > 0$$

whenever $\varphi(B) > 0$, for $B \in \mathcal{B}(\mathbb{R}^{p+q})$ and $y \in \mathbb{R}^{p+q}$.

We prove the irreducibility using results in Meyn and Tweedie (1993): if the process has a continuous component and there exists at least one reachable point in \mathbb{R}^{p+q} then the process is irreducible. In the next Proposition we show that $\underline{0}_{p+q} \in \mathbb{R}^{p+q}$ is reachable and then we prove the existence of a non-trivial continuous component under the mild assumption that the origin belongs to the interior of one of the sets R_j .

Proposition 3.1 *If there exists $\lambda \in (0, 1)$ such that*

$$|f(x_1, \dots, x_p)| \leq \lambda \max_{i=1, \dots, p} |x_i| \tag{5}$$

then the point $\underline{0}_{p+q}$ is reachable for the chain $\{Y_n\}$ defined in (3), that is, for every open set $O \in \mathcal{B}(\mathbb{R}^{p+q})$ containing $\underline{0}_{p+q}$

$$\sum_n P^n(y, O) > 0 \quad y \in \mathbb{R}^{p+q}.$$

PROOF We need to show that $\forall \xi > 0$ there exists $n \geq 1$ such that

$$\Pr[\|Y_n\| < \xi | Y_0 = y] > 0$$

with $y \in \mathbb{R}^{p+q}$. Since the number m of different regimes and the orders q are finite, then $b = 1 + \max_{i,j} |b_i^{(j)}| < +\infty$. By assumption (E) on the errors we have, for every $r > 0$,

$$\Pr[e_n \in (-r, r)] > 0. \quad (6)$$

By (6) we have that $\forall \varepsilon > 0$ the error sequence is such that

$$\Pr\left[|e_1| < \frac{\varepsilon \delta}{2b(q+1)}\right] > 0, \quad (7)$$

with $\delta \in (0, 1)$ to be determined in the following. Therefore, from now on, we assume that, for every $n \geq 1$,

$$|e_n| < \delta \frac{\varepsilon}{2b(q+1)} < \frac{\varepsilon}{2}. \quad (8)$$

By (5) and (8) we obtain that for $n > q$

$$\begin{aligned} |X_n| &\leq \lambda \max_{i=1, \dots, p} |X_{n-i}| + b \sum_{i=1}^q |e_{n-i}| + b|e_n| \\ &= \lambda \max_{i=1, \dots, p} |X_{n-i}| + \Delta_1, \end{aligned} \quad (9)$$

with $\Delta_1 = \delta \frac{\varepsilon}{2}$. Iterating, we obtain

$$|X_{n+1}| \leq \lambda \max\{|X_{n-1}|, \dots, |X_{n-p+1}|, \lambda |X_{n-p}|\} + \Delta_1(1 + \lambda),$$

and

$$|X_{n+p}| \leq \lambda^2 \max_{i=1, \dots, p} |X_{n-i}| + \Delta_1 \frac{1 - \lambda^{p+1}}{1 - \lambda}.$$

In the same way we have for $k = 0, 1, \dots$,

$$|X_{n+kp}| \leq \lambda^{k+1} \max_{i=1, \dots, p} |X_{n-i}| + \Delta_1 \frac{1 - \lambda^{kp+1}}{1 - \lambda}. \quad (10)$$

Summarizing, by (10), we obtain that

$$|X_{n+kp}| \leq \lambda^{k+1} \max_{i=1, \dots, p} |X_{n-i}| + \delta \frac{\varepsilon}{2(1 - \lambda)} \quad (11)$$

and, for $i = 1, \dots, p$,

$$\begin{aligned} |X_{n+kp+i}| &\leq \lambda \max\{\lambda^k |X_{n-1}|, \dots, \lambda^k |X_{n-p+i+1}|, \lambda^{k+1} |X_{n-p+i}|, \dots, \lambda^{k+1} |X_{n-p}|\} + \\ &+ \Delta_1 \frac{1 - \lambda^{kp+i+1}}{1 - \lambda} \leq \lambda^{k+1} \max_{i=1, \dots, p} |X_{n-i}| + \delta \frac{\varepsilon}{2(1 - \lambda)}. \end{aligned} \quad (12)$$

Therefore, by (11) and (12),

$$\exists \delta < \min\{1 - \lambda, b(q+1)\} : \forall Y_0 = y \in \mathbb{R}^{p+q} \quad \exists k < \infty : |X_{q+1+kp+i}| < \frac{\varepsilon}{2} \quad \forall i = 1, \dots, p.$$

This implies, by (9) and (8), that there exists $n < \infty$ such that

$$|X_n| \leq \lambda \max_{i=1, \dots, p} |X_{n-i}| + \delta \frac{\varepsilon}{2} \leq (\lambda + \delta) \frac{\varepsilon}{2} < \varepsilon$$

and therefore all the components of Y_n are smaller than ε . By (7) now we conclude that for every $\varepsilon > 0$

$$\Pr[\|Y_n\| < c\varepsilon | Y_0 = y] > 0$$

with $c > 0$ constant not depending on ε and the proof is complete. \blacksquare

Before to prove the main result of this Section we need to state two basic assumptions on the model.

Let $R_{j_0}^* \subset \mathbb{R}^{p+q}$ be the element of the partition that contains $\underline{0}_{p+q}$. In the following we assume that

(T) there exists $\epsilon > 0$ such that $(-\epsilon, \epsilon)^{p+q} \subset R_{j_0}^*$, that is, $\underline{0}_p$ belongs to the interior of R_{j_0} .

Let $f_0 : R_{j_0} \rightarrow \mathbb{R}$ be the restriction of the function $f(\cdot)$ in (1) to the set R_{j_0} and $Y_{n,0}$ be the Markovian representation of the model with $f_0(\cdot)$ in place of $f(\cdot)$. Then we assume that

(F) $Y_{n,0}$ is a T-chain.

Theorem 3.1 *The Markov chain Y_n defined in (3), with assumptions (E), (T) and (F) is a φ -irreducible and T-continuous process provided (5) is satisfied.*

PROOF We proved that the chain reaches with positive probability a neighborhood of $\underline{0}_{p+q}$ from every starting point. By assumption (E), which leads to (6), we have that for fixed $n^* < \infty$ and $0 < \delta_1 < \frac{\epsilon}{2}$ there exists $\delta_2 > 0$ such that $\Pr[Y_{n^*} \in R_{j_0} | Y_0 = y] > 0$ whenever $|e_n| < \delta_2$ for every $n \leq n^*$ and $Y_0 \in (-\delta_1, \delta_1)^{p+q} = I_{p+q}$. Therefore we consider in the following $Y_0 = y \in I_{p+q}$. Moreover, setting $J_{p+q} = (-\delta_2, \delta_2)^{p+q}$ and $e^{p+q} = (e_1, \dots, e_{p+q})'$, we have for $B \in \mathcal{B}(\mathbb{R}^{p+q})$

$$\begin{aligned} \Pr[Y_{p+q} \in B | Y_0 = y] &= \Pr[Y_{p+q} \in B | Y_0 = y, e^{p+q} \in J_{p+q}] \Pr[e^{p+q} \in J_{p+q}] + \\ &\quad + \Pr[Y_{p+q} \in B | Y_0 = y, e^{p+q} \in J_{p+q}^c] \Pr[e^{p+q} \in J_{p+q}^c] \end{aligned}$$

and $\Pr[e^{p+q} \in J_{p+q}] > 0$. Hence, if

$$P^*(y, B) = \Pr[Y_{p+q} \in B | Y_0 = y, e^{p+q} \in J_{p+q}] \tag{13}$$

has a non-trivial continuous component then P^{p+q} possesses a non-trivial continuous component as well and Y_n is a T-chain. Therefore in the following we assume $(e_1, \dots, e_{p+q})' \in J_{p+q}$. Note that since the chain is not leaving $R_{j_0}^*$ in the first $p+q$ steps then by the hypothesis on function $f(\cdot)$ the model is behaving as a T-continuous process. Therefore by assumption (E) and Proposition 6.3.3 in Meyn and Tweedie (1993) P^* defined in (13) has a non-trivial continuous component and hence $\{Y_n\}$ is a T-chain. Since by Proposition 4.1 there exists at least one *reachable* point, i.e. $\underline{0}_{p+q}$, $\{Y_n\}$ is φ -irreducible by Proposition 6.2.1 in Meyn and Tweedie (1993). The irreducibility measure φ is the continuous component of the transition kernel at $\underline{0}_{p+q}$. \blacksquare

Remark 3.1 *If assumption (T) is not satisfied then $\underline{0}_p \in \partial R_j$ for at least two different sets R_j , $j \in \{1, \dots, m\}$. Sufficient conditions for the irreducibility might be obtained using the technique outlined in Cline and Pu (1999) but taking into account only the regions R_j that have non empty intersections with every neighborhood of the origin. The drawback is that, unless the model is very simple, it could be very complicate to obtain explicit conditions on the coefficients.*

Corollary 3.1 *The Threshold ARMA process $\{X_n\}$ defined in (2) with assumptions (E) and (T) has an irreducible and T-continuous Markovian representation provided that*

$$\max_{j=1, \dots, m} \sum_{i=1}^p |a_i^{(j)}| < 1. \quad (14)$$

PROOF If the function $f(\cdot)$ in (1) is

$$f(\underline{X}_{n-1}) = \sum_{j=1}^m \left(\sum_{i=1}^p a_i^{(j)} X_{n-i} \right) \mathbb{1}_{R_j}(\underline{X}_{n-1})$$

then we obtain the model defined in (2). If $\lambda = \max_{j=1, \dots, m} \sum_{i=1}^p |a_i^{(j)}|$ we have

$$\begin{aligned} |f(x_1, \dots, x_p)| &\leq \max_{i=1, \dots, p} |x_i| \sum_{j=1}^m \sum_{i=1}^p |a_i^{(j)}| \mathbb{1}_{R_j}((x_1, \dots, x_p)') \\ &\leq \max_{i=1, \dots, p} |x_i| \max_{j=1, \dots, m} \sum_{i=1}^p |a_i^{(j)}| = \lambda \max_{i=1, \dots, p} |x_i| \end{aligned}$$

and (5) is satisfied. Therefore if (14) holds we can apply Proposition 3.1 and Theorem 3.1. Indeed note that the model (2) restricted to $R_{j_0}^*$ is behaving as a linear ARMA process and its controllability matrix (see Meyn and Tweedie (1993), pag. 95) has rank $p+q$ because the ARMA regimes are in reduced form as assumed at the beginning of Section 1. Therefore assumption (F) is satisfied and the proof is complete. ■

4 Geometric ergodicity of the model

Since we have proved the φ -irreducibility and T-continuity of the Markov chain $\{Y_n\}$ we can now state the following

Theorem 4.1 *The process $\{X_n\}$ defined in (1) with the assumption (E), (T) and (F) is geometrically ergodic provided that (5) is satisfied.*

Remark 4.1 *This last result is in accordance with Chan and Tong (1985) who state the same sufficient condition on the coefficients for the ergodicity of a SETAR($m; p$) process.*

Moreover An and Huang (1996) use a condition similar to (5) to prove the geometric ergodicity of nonlinear autoregressive models.

PROOF By Theorem 15.0.1 in Meyn and Tweedie (1993), we need a non-negative, measurable and locally bounded function $V(\cdot) : \mathbb{R}^{p+q} \rightarrow \mathbb{R}$ which satisfies, for any $y \in \mathbb{R}^{p+q}$,

$$\mathbb{E}[V(Y_n)|Y_{n-1} = y] \leq \delta V(y) + c \quad (15)$$

for some $\delta \in (0, 1)$ and $c > 0$. To achieve our goal we define, as in Cline and Pu (1999),

$$V(Y_n) = v(\underline{X}_n) + \sum_{i=0}^{q-1} L_i |e_{n-i}| \quad (16)$$

with

$$v(\underline{X}_n) = \max_{i=0, \dots, p-1} (\rho^i |X_{n-i}|)$$

and, by (5),

$$\rho = (\lambda)^{\frac{1}{p}} < 1. \quad (17)$$

The constants $L_i, i = 0, \dots, q-1$, are determined in the following.

For $y = (y_1, \dots, y_{p+q})' \in \mathbb{R}^{p+q}$ we have

$$\begin{aligned} \mathbb{E}[V(Y_n)|Y_{n-1} = y] &= \mathbb{E}[v(\underline{X}_n)|Y_{n-1} = y] + \mathbb{E} \left[\sum_{i=0}^{q-1} L_i |e_{n-i}| \mid Y_{n-1} = y \right] \\ &= \mathbb{E}[v(\underline{X}_n)|Y_{n-1} = y] + L_0 M + \sum_{i=1}^{q-1} L_i |y_{p+i}| \end{aligned} \quad (18)$$

because the error sequence is composed by i.i.d. random variables. Since the number of different regimes m in (2) is finite and for each of those the q coefficients of the MA part are real constants, we have $\max_{i,j} |b_i^{(j)}| < +\infty$. This implies that $\exists b > 0$ such that, using (5) and (17), we have for any $j \in \{1, \dots, m\}$

$$\begin{aligned} |X_n| &\leq |f(\underline{X}_{n-1})| + b \sum_{i=1}^q |e_{n-i}| + |e_n| \leq \rho^p \max_{i=1, \dots, p} |X_{n-i}| + |e_n| + b \sum_{i=1}^q |e_{n-i}| \\ &\leq \rho \max_{i=1, \dots, p} (\rho^{i-1} |X_{n-i}|) + |e_n| + b \sum_{i=1}^q |e_{n-i}| = \rho v(\underline{X}_{n-1}) + |e_n| + b \sum_{i=1}^q |e_{n-i}|. \end{aligned}$$

Since $\rho < 1$,

$$v(\underline{X}_n) \leq \rho v(\underline{X}_{n-1}) + |e_n| + b \sum_{i=1}^q |e_{n-i}| \quad (19)$$

and, by (19),

$$\mathbb{E}[v(\underline{X}_n)|Y_{n-1} = y] \leq \rho v(\underline{x}) + M + b \sum_{i=1}^q |y_{p+i}| \quad (20)$$

with $\underline{x} = (y_1, \dots, y_p)'$. By (20) and (18) we have

$$\mathbb{E}[V(Y_n)|Y_{n-1} = y] \leq \rho v(\underline{x}) + (L_0 + 1)M + \sum_{i=1}^{q-1} (L_i + b) |y_{p+i}| + b |y_{p+q}| \quad (21)$$

and by (16) we obtain

$$\rho V(y) = \rho v(\underline{x}) + \sum_{i=1}^q \rho L_{i-1} |y_{p+i}|. \quad (22)$$

Comparing (21) and (22) we obtain that (15) is satisfied with $\delta = \rho$, $c = (L_0 + 1)M$ and the constants L_i , $i = 0, \dots, q-1$, such that $\rho L_{q-1} \geq b$ and $\rho L_{i-1} \geq L_i + b$ for $i = 1, \dots, q-1$. Since $\rho < 1$ and $M = \mathbb{E}[|e_1|] < \infty$ the proof is complete. ■

Remark 4.2 Above we have used the functions described in the proofs of Theorem 3.1 and Lemma 6.1 in Cline and Pu (1999) to obtain the suitable one-step inequality (15).

Corollary 4.1 The process $\{X_n\}$ defined in (2) with assumptions (E) and (T) is geometrically ergodic provided that (14) holds.

Acknowledgments

The author would like to thank Peter Brockwell for introducing me to the problem and Richard Tweedie for the useful discussions, Daren Cline and Marco Ferrante for pointing out some problems in an early draft of the paper and an unknown referee for the useful suggestions.

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