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A Morse-type index for critical points of vector functions

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Abstract

In this work we study the critical points of vector functions form $\mathbb{R}^n$ to $\mathbb{R}^m$ with $n \geq m$, following the definition introduced by S. Smale in the context of vector optimization. The local monotonicity properties of a vector function around a critical point which are invariant with respect to local coordinate changes are considered. We propose a classification of critical points through the introduction of an index for a critical point consisting of a triple of nonnegative integers. The proposed index is based on the "sign" of an appropriate vector-valued second-order differential, that is proved to be invariant with respect to local coordinate changes. In order to avoid anomalous behaviours of the Jacobian matrix, the analysis is partially restricted to the proper critical points, a subset of critical points which enjoy stability properties with respect to perturbations of the order structure. Under non-degeneracy conditions, the index is proved to be locally constant. Moreover, the stability properties of the index with respect to perturbations both of the ordering cone and of the function are considered. Finally, the consistency of the proposed classification with the one given by Whitney for stable maps from the plane into the plane is proved.

1 Introduction

When we consider a C^2 function $f : \mathbb{R}^n \to \mathbb{R}$, a critical or stationary point x^0 is a point where the gradient vanishes, that is a point which satisfies a necessary optimality condition for the problem of finding the local minima of f.

The extension of such a notion to a vector function $f : \mathbb{R}^n \to \mathbb{R}^m$ is not univoquely determined. Indeed, the notion of critical point of a vector function can be introduced following two different approaches. The first one is developed in the framework of the theory of singularities, where the space $\mathbb{R}^m$ is not endowed with an order structure. In this context a point x^0 is called critical (or singular) when the Jacobian of f at x^0 has not full rank. The study of singularities of

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differentiable maps was developed in the 60's and 70's; for a detailed reference on the topic one can see, e.g., the books [1] and [2].

An alternative approach for defining a notion of criticality for vector functions is strongly related to the theory of vector optimization, where the space $\mathbb{R}^m$ is partially ordered by a closed convex pointed cone with nonempty interior. In this approach, which was introduced in the pioneering works by Smale [21], [23], a critical point is a point that satisfies the classical first order necessary optimality condition for a weakly efficient solution for a vector optimization problem. It is easily shown that the notion of criticality defined through the second approach is a restriction of the classical notion of singularity of a vector function introduced in the first approach. In this paper we follow mainly the second approach. When we refer to a point where the Jacobian matrix has not full rank, we will use the term "singular point", while we will name "critical point" a point that satisfies the first order necessary condition for a vector optimization problem. The study of critical points in the context of vector optimization was introduced by Smale [21], [22], [23], [24]. Further attention to the topic was given by Wan [25], [26] and Marzollo, Pascoletti, Serafini [12], [13]. Some recent contributions can be found in [4], [10], [15].

In the scalar case, the classical Morse index of a stationary point x^0 of a function f is introduced under nondegeneracy conditions, i.e. under the assumption that the Hessian matrix of f at x^0 has no zero eigenvalues. The Morse index is defined as the index of the bilinear form obtained from the second order differential of f at the considered point. The invariance of the Morse index comes out from the invariance, with respect to local coordinate changes, of the sign of the eigenvalues of the quadratic form associated to the second order differential at the stationary points, see e.g. [8].

The aim of our paper is to introduce an index that summarizes the local monotonicity properties of f around a critical point which are invariant to local coordinate changes. In the vector case, the definition of an invariant index for a critical point should therefore be preceded by the introduction of a suitable notion of an invariant (vector) second order differential that has a similar role to the so-called "quadratic differential", which was introduced by Porteus [18] and Mather [14] within the singularity theory for maps. The proposed index consists of a triple of nonnegative integer numbers. This triple is linked to the "sign" of the proposed second order differential. We prove that, in the special case $m = 1$, the given index coincides with the classical Morse index.

Through the proposed index we obtain a classification of the critical points of a vector-valued function. Following the proposed classification, our aim is to obtain detailed information about the descent directions starting from a critical point, extending to the vector-valued case some results concerning saddle points which are well known in the scalar case. In the study of the links between the values of the index and the local monotonicity properties of f we need to avoid some anomalous behaviours of the Jacobian matrix. This leads to consider a subset of the

critical points of f that we name "proper critical points". The choice of this name is motivated since such points enjoy stability properties with respect to perturbations of the order structure similar to those of properly efficient points.
A fundamental result in Morse Theory states that nondegenerate critical points are isolated. In the vector case critical points are typically non-isolated, even under nondegeneracy assumptions. However, we can prove that, under nondegeneracy conditions the index of a critical point is locally constant. Further, we develop a study of the stability properties of the index with regard to perturbations both of the ordering cone and of the function f. This study is carried out in the case where the Jacobian matrix of f at a critical point x^0 has corank one. The relevance of this special case was highlighted in some papers by Smale [21], [23]. Finally we show that our classification of critical points is consistent with the classical results obtained by Whitney [27] for stable maps from the plane to the plane.

2 Preliminaries

Let $f = (f_1, \dots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$, $n \geq m$, $f \in C^2(\mathbb{R}^n)$ and $P \subseteq \mathbb{R}^m$ a closed pointed (i.e. $P \cap (-P) = 0$) convex cone with nonempty interior. We consider a vector $x \in \mathbb{R}^k$ as a $k \times 1$ matrix and we denote by B the closed unit ball in $\mathbb{R}^n$. We consider $\mathbb{R}^m$ partially ordered by P in the following way:

$$y \leq_P x \ \text{ if and only if } \ y - x \in -P\,.$$

In the sequel we consider polyhedral ordering cones. We recall that a cone $P \subseteq \mathbb{R}^m$ is said to be *polyhedral* when there exist s vectors $a^i \in \mathbb{R}^m$, $i = 1, \dots, s$, such that $P = \text{co conv}\,\{a^1, \dots, a^s\}$ (here "co" stands for the cone generated and "conv" for the convex hull) . In this paper we assume that P is a pointed polyhedral cone with r extremal rays. We recall that an extremal ray of P is an half-line R such that $x, y \in P$ and $tx + (1-t)y \in R$ for some $t \in (0,1)$ entails $x, y \in R$. If P has r extremal rays, it is possible to find r vectors $a^1, \dots, a^r$, such that $P = \text{co conv}\,\{a^1, \dots, a^r\}$ and r is the minimum number of vectors such that this equality holds. Clearly each vector a^i, $i = 1, \dots, r$ lays on an extremal ray of P. We refer to these vectors as the generators of P. Assuming, without loss of generality, $\|a^i\| = 1\,, i = 1, \dots, r$, these vectors can be thought uniquely determined.

We denote with $P' = \{y' \in \mathbb{R}^m : \langle y', y\rangle \geq 0,\ \forall y \in P\}$, the *positive polar* cone of P. It is known that if P is polyhedral, so is also P'. We denote by e the number of extremal rays of P'. We consider the following vector optimization problem:

$$\min_P \ f(x) \ \ x \in \mathbb{R}^n\,. \tag{1}$$

We recall that a point $x^0 \in \mathbb{R}^n$ is a *locally weakly efficient* solution (*locally efficient solution*) of problem (1) when there exists a neighborhood U of x^0 such that $f(x) - f(x^0) \notin -\text{int}\,P$

$(f(x) - f(x^0) \notin -P\backslash\{0\})$ for every $x \in U$. A point $x^0 \in \mathbb{R}^n$ is a *locally ideal solution* of problem (1) when $f(x) - f(x^0) \in P$ for every $x \in U$. We denote respectively with WEff (f, P), Eff (f, P), IEff (f, P) the set of locally weakly efficient solutions, locally efficient solutions and locally ideal solutions of problem (1). In the sequel we omit the word "locally". For a deeper exposition of vector optimization theory see e.g. [3], [7], [9].

The classical notion of singularity for vector valued functions is the following.

Definition 2.1. *A point $x \in \mathbb{R}^n$ is called a singular point when $f'(x)$ has not full rank.*

In the present work we focus on a more restrictive notion of critical point which is suitable for applications to vector optimization. This notion has been introduced by Smale in [21], [23].

Definition 2.2. *We define the set of critical points of f as:*

$$K(f, P) = \{x \in \mathbb{R}^n : f'(x)u \notin -\mathrm{int}P, \quad \forall u \in \mathbb{R}^n\}.$$

The following proposition characterizes the critical points of the function f.

Proposition 2.1. *[23] The point x is critical if and only if there exists $\lambda \in P'\backslash\{0\}$ such that $\lambda^T f'(x) = 0$.*

Remark 2.1. *i) From the previous proposition it follows immediately that every critical point is also a singular point.*

ii) Let x be a singular point of f. Then there exists a nonzero vector $\lambda = \lambda(x) \in \mathbb{R}^m$ such that $\lambda^T f'(x) = 0$. If we consider the set $N_\lambda(x) = \{y \in \mathbb{R}^m, \lambda^T y \geq 0\}$ and any pointed polyhedral cone with nonempty interior such that $D(x) \subseteq N_\lambda(x)$, then $x \in K(f, D(x))$. We observe that the map $\mathcal{D} : x \to D(x)$ is a domination structure (for the definition of domination structure see, e.g., [20]).

It is well known that the set of weakly efficient points is a subset of the set of critical points (see [21], [23] and [12], [13]).

Proposition 2.2. WEff$(f, P) \subseteq K(f, P)$.

We begin to study the behaviour of the set of critical points with respect to local coordinate changes in the domain and the image space. We show that the set $K(f, P)$ is invariant with respect to local coordinate changes in the domain, while a local coordinate change in the image space entails a change in the order structure in $\mathbb{R}^m$. Let U be a neighborhood of a point $x^0 \in \mathbb{R}^n$ and let $g : U \to \mathbb{R}^n$ be a C^2-diffeomorphism with $g(x^0) = x^0$. The pair (U, g) is said to be a local coordinate system around x^0. Let V be a neighborhood of $f(x^0)$ and $h : V \to \mathbb{R}^m$ be a C^2-diffeomorphism with $h(f(x^0)) = f(x^0)$. The pair (V, h) is said to be a local coordinate system around $f(x^0)$.
The proof of the next proposition is a direct consequence of the chain rule.

Proposition 2.3. *i) Let (U, g) be a local coordinate system around $x^0 \in \mathbb{R}^n$. Then $x^0 \in K(f, P)$ if and only if $x^0 \in K(f(g), P)$.*

ii) Let (V, h) be a local coordinate system around $f(x^0) \in \mathbb{R}^m$. Then $x^0 \in K(f, P)$ if and only if $x^0 \in K(h(f), h'(f(x^0))P)$.

Proof. The proof of point i) is straightforward. To prove ii), let $x^0 \in K(f, P)$. Hence, there exists $\lambda \in P' \backslash \{0\}$, such that $\lambda^T f'(x^0) = 0$. If $\tilde{P} = h'(f(x^0))P$, then $(\tilde{P})' = [h'(f(x^0))^T]^{-1} P'$. Hence, by choosing $\tilde{\lambda} = [h'(f(x^0))^T]^{-1} \lambda \in \tilde{P}'$, we obtain

$$\tilde{\lambda}^T (h(f))'(x^0) = \lambda^T h'(f(x^0))^{-1} h'(f(x^0)) f'(x^0) = 0 \,,$$

which proves that $x^0 \in K(h(f), h'(f(x^0)P)$. The converse is obvious. $\square$

Remark 2.2. *In order to preserve the property that x^0 is a critical point, when a local coordinate change is applied to the image space $\mathbb{R}^m$, the previous proposition suggests to trasform the cone P into $h'(f(x^0))P$, which clearly depends on the point x^0. Hence the order structure in the image space transforms in a domination structure.*

In the sequel, when we refer to a local coordinate change around $f(x^0)$ in the image space $\mathbb{R}^m$, we assume that also the cone P is transformed into $h'(f(x^0))P$.

3 Tools for a classification of critical points: an invariant second order differential

In the scalar case the Morse index of a stationary point x summarizes the local features of a function f around x which are invariant with respect to local coordinate changes. Such an index is defined as the index of the quadratic form represented by the Hessian matrix at x of the scalar function f and its invariance follows from the well-known invariance of the second order differential at a stationary point. The aim of this section is to construct a notion of second order differential for a vector function f which is invariant with respect to local coordinate changes both in the domain and in the image space. The construction is made by several steps and needs some preliminary notions and results.

First, we introduce the set of multipliers $\Lambda_{f,P}(x) = \{\lambda \in P' \backslash \{0\} : \lambda^T f'(x) = 0\}$. The geometrical structure of this set plays a key role in the behaviour of the proposed invariant second order differential, as shown in the sequel. From Proposition 2.1 it follows immediately that $\Lambda_{f,P}(x) \neq \emptyset$ if and only if $x \in K(f, P)$. Clearly $\Lambda_{f,P}(x)$ is itself a polyhedral cone. We denote by $l(x)$ the minimum number of generators of $\Lambda_{f,P}(x)$, that is the number of its extremal rays. Hence there exist $l(x)$ uniquely determined vectors $\lambda^1, \ldots, \lambda^{l(x)} \in \mathbb{R}^m$ of unitary norm, such that

$\Lambda_{f,P}(x) = \operatorname{co\,conv}\{\lambda^1, \dots, \lambda^{l(x)}\}$. Let $c(x) = \operatorname{corank} f'(x) = m - \operatorname{rank} f'(x)$. Since the set $\Lambda_{f,P}(x)$ is defined as the intersection of the subspace $\{\lambda \in \mathbb{R}^m : \lambda^\top f'(x) = 0\}$ of dimension $c(x)$ and the cone P', in order to achieve a deeper comprehension of the structure of the set $\Lambda_{f,P}(x)$, we need to study the relationships between $l(x)$, $c(x)$ and e (where e denotes the number of generators of P'). When $c(x) = 1$, then clearly $l(x) = 1$. This case, although simple, is meaningful and we investigate it in Section 7. Moreover, we can study the general case, using some known results. Indeed, in [16], in the context of the theory of automata, the author studies the intersection of a subspace and a polyhedral cone using the theory of convex polytopes, see, e.g., [5]. The main result in that paper allows us to give the following upper bound for $l(x)$ when $c(x) \geq 2$

$$l(x) \leq \begin{cases} \frac{2e}{c(x)-1} \begin{pmatrix} e - \frac{c(x)+1}{2} \\ \frac{c(x)-3}{2} \end{pmatrix} & \text{if c(x) is odd} \\ 2 \begin{pmatrix} e - \frac{c(x)}{2} \\ \frac{c(x)}{2} - 1 \end{pmatrix} & \text{if c(x) is even} \end{cases}$$

In the case when $c(x) = 2$, the previous estimation provides $l(x) \leq 2 \leq e$, so that the upper bound does not depend on e. Further, when $c(x) = 3$, we get $l(x) \leq e$. Surprisingly when $c(x) > 3$ one could have $l(x) > e$ (see e.g. [16]).

Now we investigate the behaviour of $\Lambda_{f,P}(x)$ with respect to local coordinate changes both in the domain and the image space.

Proposition 3.1. *Let $x^0 \in K(f, P)$.*

i) $\Lambda_{f,P}(x^0)$ does not depend on the local coordinate system around $x^0 \in \mathbb{R}^n$.

ii) When a local coordinate change h around $f(x^0)$ is applied to the image space $\mathbb{R}^m$, then $\Lambda_{f,P}(x^0)$ tranforms into

$$\Lambda_{h(f),\, h'(f(x^0))P}(x^0) = [h'(f(x^0))^T]^{-1} \Lambda_{f,P}(x^0)$$

and hence $\Lambda_{h(f),\, h'(f(x^0))P}(x^0) = \operatorname{co\,conv}\{\tilde{\lambda}^1, \dots, \tilde{\lambda}^{l(x^0)}\}$, where $\tilde{\lambda}^i = [h'(f(x^0))^T]^{-1}\lambda^i$, with $\lambda^1, \dots, \lambda^{l(x)}$ such that $\Lambda_{f,P}(x) = \operatorname{co\,conv}\{\lambda^1, \dots, \lambda^{l(x)}\}$.

Proof. i) In the new local coordinate system (U, g), we have

$$\Lambda_{f,P}(x^0) = \{\lambda \in P' \backslash \{0\} : \lambda^T f'(x^0)(g^{-1})'(y^0) = 0\},$$

where $y^0 = f(x^0)$. Since $(g^{-1})'(y^0)$ is nonsingular, the thesis easily follows.

ii) When we apply the coordinate change h, P transforms into $h'(f(x^0))P$ and so P' transforms into $[h'(f(x^0))^T]^{-1}P'$. Hence $\Lambda_{f,P}(x^0)$ transforms into

$$\Lambda_{h(f),\, h'(f(x^0))P}(x^0) = \{\tilde{\lambda} \in [h'(f(x^0))^T]^{-1}P' : \tilde{\lambda}^T h'(f(x^0))f'(x^0) = 0\}$$

$$= [h'(f(x^0))^T]^{-1}\Lambda_{f,P}(x^0).$$

$\square$

We denote by $f_i''(x)$ the Hessian of $f_i : \mathbb{R}^n \to \mathbb{R}$ at the point x. With $f''(x)$ we denote the vector of matrices $(f_1''(x), \ldots, f_m''(x))^T$. The bilinear map $f''(x)(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$f''(x)(u, v) = (v^T f_1''(x)u, \ldots, v^T f_m''(x)u)^T$$

does not make sense invariantly with respect to local coordinate changes. Nevertheless, the function $f''(x)(\cdot,\cdot)$ induces a map

$$H_f(x) : \mathbb{R}^n \times \ker f'(x) \to \operatorname{coker} f'(x)\,,$$

where $\operatorname{coker} f'(x) = \mathbb{R}^m/\operatorname{im} f'(x)$. This map is called the *second intrinsic derivative* of the map f at x and is invariantly defined with respect to local coordinate changes [14]. Now we consider the map $H_f(x)(u, u)$ from $\ker f'(x)$ to $\operatorname{coker} f'(x)$ which is called the quadratic differential of f at x (see, e.g., [1]). From the definition it follows that

$$H_f(x)(u, u) = [f''(x)(u, u)]_{\operatorname{im} f'(x)}\,,$$

for every $u \in \ker f'(x)$, where $[f''(x)(u, u)]_{\operatorname{im} f'(x)}$ denotes the class of equivalence in the quotient space $\mathbb{R}^m/\operatorname{im} f'(x)$, i.e. $[f''(x)(u, u)]_{\operatorname{im} f'(x)} = f''(x)(u, u) + \operatorname{im} f'(x)$. For brevity, we will denote $H_f(x)(u, u)$ and $f''(x)(u, u)$ by $H_f(x)(u)$ and $f''(x)(u)$.

Now we introduce a modified version of the quadratic differential. This notion will play a key role in the definition of an index for a critical point. We define a function $H_{\Lambda,f}(x) : \ker f'(x) \to \mathbb{R}^{l(x)}$ as

$$H_{\Lambda,f}(x)(u) = ((\lambda^1)^T H_f(x)(u), \ldots, (\lambda^{l(x)})^T H_f(x)(u))\,, \tag{2}$$

where $\lambda^1, \ldots, \lambda^{l(x)}$ are vectors of unit norm which generate the cone $\Lambda_{f,P}(x)$ (clearly these vectors depend on the point x). In (2), for the sake of simplicity, we denote $\Lambda_{f,P}(x^0)$ by Λ. We use this notation to underline that this quadratic differential depends on the structure of $\Lambda_{f,P}(x^0)$. Let $x \in K(f, P)$. From the definition of $H_f(x)(u)$, we have

$$H_{\Lambda,f}(x)(u) = ((\lambda^1)^T f''(x)(u), \ldots, (\lambda^{l(x)})^T f''(x)(u))\,.$$

Preliminarly we prove the invariance properties of $H_{\Lambda,f}(x)$ with respect to local coordinate changes in the domain and the image space.

Proposition 3.2. *Let $x \in K(f, P)$.*

i) $H_{\Lambda,f}(x)$ is invariant on $\ker f'(x)$ with respect to local coordinate changes in the domain.

ii) $H_{\Lambda,f}(x)$ is invariant on $\ker f'(x)$, with respect to local coordinate changes in the image space.

Proof. i) The thesis follows immediately, since $H_f(x)$ is invariant to local coordinate changes on $\ker f'(x)$.

ii) Let $h = (h_1, \dots, h_m) : \mathbb{R}^m \to \mathbb{R}^m$ be a local coordinate change in the image space and let $\psi(x) = h(f(x))$. Hence $\psi : \mathbb{R}^n \to \mathbb{R}^m$ and we have

$$\psi'(x) = h'(f(x))f'(x)$$

and

$$\begin{aligned}\psi''(x)(u) &= h''(f(x))(f'(x)(u)) + h'(f(x))(f''(x)(u)) \\ &= \begin{pmatrix} u^T[f'(x)]^T h_1''(f(x))f'(x)u + h_1'(f(x))f''(x)(u) \\ \cdots \\ u^T[f'(x)]^T h_m''(f(x))f'(x)u + h_m'(f(x))f''(x)(u) \end{pmatrix}\end{aligned}$$

(see e.g. Theorem 10, Chapter 6 in [11]). When $u \in \ker f'(x)$, we have

$$\psi''(x)(u) = \begin{pmatrix} h_1'(f(x))f''(x)(u) \\ \cdots \\ h_m'(f(x))f''(x)(u) \end{pmatrix}$$

and hence

$$\begin{aligned} H_{\tilde{\Lambda},\psi}(x)(u) &= ((\tilde{\lambda}^1)^T H_\psi(x)(u), \dots, (\tilde{\lambda}^{l(x)})^T H_\psi(x)(u))^T \\ &= ((\tilde{\lambda}^1)^T \psi''(x)(u), \dots, (\tilde{\lambda}^{l(x)})^T \psi''(x)(u)), \end{aligned}$$

where $\tilde{\Lambda} = \Lambda_{h(f),\, h'f(x^0)P}(x^0)$ and $\tilde{\lambda}^i = [h'(f(x^0))^T]^{-1}\lambda^i$ (see Proposition 3.1). Hence

$$H_{\tilde{\Lambda},\psi}(x)(u) = ((\lambda^1)^T f''(x)(u), \dots, (\lambda^{l(x)})^T f''(x)(u)) = H_{\Lambda,f}(x)(u)\,.$$

□

As it is easily seen from the definitions, $H_f(x)(u)$ and $H_{\Lambda,f}(x)(u)$ are strictly related. Indeed, when $c(x) = 1$, (and hence also $l(x) = 1$), Smale proves that $H_f(x)(u)$ can be represented as $H_{\Lambda,f}(x)(u)$ (see "2nd order proposition" in [23]).
The next result extends this representation to the case where $c(x) = l(x)$.

Proposition 3.3. *If $c(x) = l(x)$, then $H_f(x)(u)$ can be represented as $H_{\Lambda,f}(x)(u)$.*

Proof. First we observe that the same value $H_{\Lambda,f}(x)(u)$ corresponds to a pair of representative elements of the same equivalence class $[f''(x)(u)]_{\operatorname{im} f'(x)}$ in $\mathbb{R}^m/\operatorname{im} f'(x)$. We have only to show that to every value $H_{\Lambda,f}(x)(u)$ corresponds a unique equivalence class $[f''(x)(u)]_{\operatorname{im} f'(x)}$. Indeed, let us observe that since the cone $\Lambda_{f,P}(x)$ has $l(x)$ extremal rays, the generators $\lambda^1, \dots, \lambda^{l(x)}$ are linearly independent, since $\Lambda_{f,P}(x)$ is cointained in a subspace of dimension $l(x)$. Hence these vectors are a base for the linear subspace orthogonal to $\operatorname{im} f'(x)$, i.e. $(\operatorname{im} f'(x))^\perp$. By contradiction, assume that there exist two different equivalence classes $[f''(x)(u^1)]_{\operatorname{im} f'(x)}$ and $[f''(x)(u^2)]_{\operatorname{im} f'(x)}$ with $H_{\Lambda,f}(x)(u^1) = H_{\Lambda,f}(x)(u^2)$. Let y and z two representative elements of $[f''(x)(u^1)]$ and $[f''(x)(u^2)]_{\operatorname{im} f'(x)}$ respectively. We have

$$((\lambda^1)^T y, \dots, (\lambda^{l(x)})^T y) = ((\lambda^1)^T z, \dots, (\lambda^{l(x)})^T z)\,,$$

that is

$$(\lambda^1)^T (y - z) = \cdots = (\lambda^{l(x)})^T (y - z) = 0\,.$$

Since the vectors $\lambda^1, \dots, \lambda^{l(x)}$ are a base of $(\operatorname{im} f'(x))^\perp$, we obtain $y - z \in \operatorname{im} f'(x)$, a contradiction. $\square$

The proof of Proposition 3.3 clarifies that the previous representation does not hold when $c(x) \neq l(x)$. Indeed, in this case, the vectors $\lambda^1, \dots, \lambda^{l(x)}$ do not constitute a base for $(\operatorname{im} f'(x))^\perp$ and hence to different equivalence classes $[f''(x)(u^1)]_{\operatorname{im} f'(x)}$ and $[f''(x)(u^2)]_{\operatorname{im} f'(x)}$ can correspond the same value of $H_{\Lambda,f}(x)(\cdot)$.

4 An index for a critical point

Using the notion of second differential introduced in the previous section, we define an index for a critical point. This index summarizes those local monotonicity features of a vector function (with respect to the order induced by the cone P) which are invariant to local coordinate changes in the domain and the image space.

Let $x^0 \in K(f,P)$. In the sequel we consider the second order differential $H_{\Lambda,f}(x^0)(u)$ only along the directions $u \in \ker f'(x^0)$. Indeed, along the directions outside $\ker f'(x^0)$, the local behaviour of the function f near the critical point x^0 is completely determined by $f'(x^0)$ (see Proposition 5.1 below). In order to construct an index for a critical point x^0, we divide the directions $u \in \ker f'(x^0)$ into three sets:

$$C^+_{f,P}(x^0) = \{u \in \ker f'(x^0) : H_{\Lambda,f}(x^0)(u) \in \operatorname{int} \mathbb{R}^{l(x^0)}_+\} \cup \{0\},$$

$$C^-_{f,P}(x^0) = \{u \in \ker f'(x^0) : H_{\Lambda,f}(x^0)(u) \in -\operatorname{int} \mathbb{R}^{l(x^0)}_+\} \cup \{0\},$$

$$C^\pm_{f,P}(x^0) = [\ker f'(x^0) \backslash (C^-_{f,P}(x^0) \cup C^+_{f,P}(x^0))] \cup \{0\}\,.$$

It is easy to verify that the previous sets are cones. Moreover the sets $C^+_{f,P}(x^0)\backslash\{0\}$ and $C^-_{f,P}(x^0)\backslash\{0\}$ are open while $C^\pm_{f,P}(x^0)$ is closed and $C^-_{f,P}(x^0) \cup C^+_{f,P}(x^0) \cup C^\pm_{f,P}(x^0) = \ker f'(x^0)$.

Definition 4.1. *Let $x^0 \in K(f,P)$. We define the index $I_{f,P}(x^0)$ of the point x^0 as the triple $(i^+(x^0), i^-(x^0), i^\pm(x^0))$, where $i^a(x^0) = \max\{\dim H : H \subseteq C^a_{f,P}(x^0), H$ subspace of $\ker f'(x^0)\}$, $a \in \{+, -, \pm\}$.*

For the sake of simplicity, in the sequel, when it is clear which is the function and which is the cone we are referring to, we denote the index only by $I(x^0)$. In force of the behaviour of $H_{\Lambda,f}(x^0)(\cdot)$ with respect to local coordinate changes in the domain and the image space (see Proposition 3.2), we obtain the following result.

Theorem 4.1. *The index $I(x^0)$ is invariant to local coordinate changes in the domain and the image space.*

The sets $C^+_{f,P}(x^0)$, $C^-_{f,P}(x^0)$, $C^\pm_{f,P}(x^0)$ cover the whole space $\ker f'(x^0)$. A deeper study on the geometrical relationships among these cones, leads us to establish some relations among the indexes $i^+(x^0)$, $i^-(x^0)$, $i^\pm(x^0)$. We express these relations through some bounds on the sum of the triple $i^+(x^0)$, $i^-(x^0)$, $i^\pm(x^0)$. Moreover we analyze some special situations.

Theorem 4.2. *Let $x^0 \in K(f,P)$ and denote by $k(x^0)$ the dimension of $\ker f'(x^0)$.*

i) We have

$$i^+(x^0) + i^-(x^0) + i^\pm(x^0) \leq \begin{cases} \frac{3}{2}k(x^0) & \text{if } k(x^0) \text{ is even} \\ \frac{3k(x^0)-1}{2} & \text{if } k(x^0) \text{ is odd} \end{cases} . \tag{3}$$

ii) if $i^\pm(x^0) = 0$ then $i^+(x^0) = k(x^0)$ and $i^-(x^0) = 0$ or $i^+(x^0) = 0$ and $i^-(x^0) = k(x^0)$

iii) if $i^-(x^0) = 0$, then

$$i^+(x^0) + i^\pm(x^0) \leq k(x^0). \tag{4}$$

Analogously if $i^+(x^0) = 0$.

iv) if one among $i^+(x^0)$, $i^-(x^0)$ and $i^\pm(x^0)$ is equal to $k(x^0)$, then the other indexes are 0.

Proof. i) Let $c^a(x^0) = k(x^0) - i^a(x^0)$, $a \in \{+,-,\pm\}$. It is easy to see that $i^+(x^0) + i^-(x^0) + i^\pm(x^0) = 3k(x^0) - (c^+(x^0) + c^-(x^0) + c^\pm(x^0))$. We consider the integer linear programming problem

$$\max_{c^+(x^0),c^-(x^0),c^\pm(x^0)} (3k(x^0) - (c^+(x^0) + c^-(x^0) + c^\pm(x^0))) \tag{5}$$

under the constraints

$$\begin{cases} 0 \leq c^+(x^0) \leq k(x^0) \\ 0 \leq c^-(x^0) \leq k(x^0) \\ 0 \leq c^\pm(x^0) \leq k(x^0) \\ c^+(x^0) + c^-(x^0) \geq k(x^0) \\ c^+(x^0) + c^\pm(x^0) \geq k(x^0) \\ c^+(x^0) + c^\pm(x^0) \geq k(x^0) \end{cases} \tag{6}$$

We observe that the last three constraints hold since the cones $C^{a}_{f,P}(x^0)$, $a \in \{+,-,\pm\}$ intersect only at the origin.

Solving problem (5) we obtain

$$\max_{c^+(x^0),c^-(x^0),c^\pm(x^0)} (3k(x^0) - (c^+(x^0) + c^-(x^0) + c^\pm(x^0))) = \begin{cases} \frac{3}{2}k(x^0) & \text{is } k(x^0) \text{ is even} \\ \frac{3k(x^0)-1}{2} & \text{if } k(x^0) \text{ is odd} \end{cases} . \tag{7}$$

ii) Since $C^{\pm}_{f,P}(x^0) = \{0\}$, the unique way to cover the whole space $\ker f'(x^0)$ is to consider either $C^{+}_{f,P}(x^0) = \ker f'(x^0)$ or $C^{-}_{f,P}(x^0) = \ker f'(x^0)$. Indeed, the sets $C^{+}_{f,P}(x^0)\backslash\{0\}$ and $C^{-}_{f,P}(x^0)\backslash\{0\}$ are open and disjoint.

iii) The proof is straightforward solving the linear programming problem of point i) in the particular case $i^-(x^0) = 0$.

iv) Trivially one of the cones covers the whole space.

□

Remark 4.1. *After proving the existence of an upper bound for the sum of the indexes, it is natural to wonder whether a lower bound exists too. For $k(x^0) \leq 3$, one can easily prove that $i^+(x^0) + i^-(x^0) + i^\pm(x^0) \geq k(x^0)$. It remains an open question whether such a bound holds for arbitrary values of $k(x^0)$.*

Remark 4.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$, $P = \mathbb{R}_+$ and let x^0 be a nondegenerate critical point of f (i.e. $f'(x^0) = 0$ and $f''(x^0)$ has no null eigenvalues). We compare the index $I(x^0)$ with the classical Morse index (see e.g. [8]), which is clearly given by $i^-(x^0)$. In this case, we have $i^+(x^0) + i^-(x^0) = k(x^0) = n$ so that the number $i^+(x^0)$ is implicitly determined by this equality. Further, according to Morse Theory, without loss of generality, we can consider the function f, around x^0, in its canonical form*

$$f(x) = \sum_{i=1}^{i^+(x^0)} x_i^2 - \sum_{i=1}^{i^-(x^0)} x_i^2$$

and so

$$C^{\pm}_{f,P}(x^0) = \{x \in \mathbb{R}^n : \sum_{i=1}^{i^+(x^0)} x_i^2 - \sum_{i=i^+(x^0)+1}^{n} x_i^2 = 0\}$$

and it follows that $i^\pm(x^0) = \min\{i^-(x^0), i^+(x^0)\}$. Indeed, since two hyperplanes of dimension $i^+(x^0)$ and $i^-(x^0)$ contained respectively in $C^{+}_{f,P}(x^0)$ and $C^{-}_{f,P}(x^0)$ generate $\mathbb{R}^n$ as direct sum, it follows $i^\pm(x^0) \leq \min\{i^-(x^0), i^+(x^0)\}$. To show that $i^\pm(x^0) \geq \min\{i^-(x^0), i^+(x^0)\}$, suppose, *without loss of generality, that* $i^+(x^0) < i^-(x^0)$*. It is enough to observe that the* $i^+(x^0)$ *vectors*

$$(\underbrace{1,0,\ldots,0}_{i^+(x^0)},\underbrace{1,0,\ldots,0}_{i^-(x^0)})$$
$$(\underbrace{0,1,\ldots,0}_{i^+(x^0)},\underbrace{0,1,\ldots,0}_{i^-(x^0)})$$
$$\ldots$$
$$(\underbrace{0,0,\ldots,1}_{i^+(x^0)},\underbrace{0,0,\ldots 1,\ldots,0}_{i^-(x^0)})$$

are linearly independent and belong to $C^{\pm}_{f,P}(x^0)$*.*

5 The behaviour of f around critical points

The results obtained in this section allow us to study the relationships among the values of the index $I(x^0)$ and the monotonicity properties of f with respect to the ordering cone, around a critical point x^0. These monotonicity properties play an important role in the study of the local solutions of the vector optimization problem (1). Moreover, these results allow us to single out the critical points with saddle behaviour. Indeed, we prove that when $i^+(x^0) \neq 0$ and $i^-(x^0) \neq 0$, f admits both non-increase and non-decrease directions starting at x^0. We also prove the existence of quadratic curves starting at x^0 along which f is strictly increasing and and of quadratic curves where f is strictly decreasing.

Theorem 5.1. *Let* $u \in C^-_{f,P}(x^0)$*,* $u \neq 0$*.*

- *i) There exists a positive number* ε *such that for every nonzero* $\delta \in (-\varepsilon,\varepsilon)$ *and for every* $u' \in u + \varepsilon B$ *it holds* $f(x^0 + \delta u') - f(x^0) \notin P$*.*
- *ii) There exist a vector* $w \in \mathbb{R}^n$ *and a positive number* ε *such that for every nonzero* $\delta \in (-\varepsilon,\varepsilon)$ *and* $w' \in w + \varepsilon B$*, it holds* $f(x^0 + \delta u + \frac{\delta^2}{2} w') - f(x^0) \in -\operatorname{int} P$*.*

Proof. i) Since $u \in C^-_{f,P}(x^0)$, for every $\lambda \in \Lambda_{f,P}(x^0)$ such that $\lambda^T f''(x^0)(u) < 0$, from Taylor's formula we obtain the existence of a positive number ε such that for every nonzero $\delta \in (-\varepsilon,\varepsilon)$ and for every $u' \in u + \varepsilon B$, we have $\lambda^T(f(x^0 + \delta u') - f(x^0)) < 0$. It follows that $f(x^0 + \delta u') - f(x^0) \notin P$.

ii) If $u \in C^-_{f,P}(x^0)$, then

$$(\operatorname{im} f'(x^0) + f''(x^0)(u)) \cap (-\operatorname{int} P) \neq \emptyset\,. \tag{8}$$

Indeed, if the previous intersection is empty, then there exists $\lambda \in P'$, $\lambda \neq 0$, such that

$$\lambda^T h \geq 0,\ \forall h \in \operatorname{im} f'(x^0) + f''(x^0)(u)\,.$$

From this inequality we obtain $\lambda^T f'(x^0) = 0$ and $\lambda^T f''(x^0)(u) \geq 0$, which contradicts $u \in C^-_{f,P}(x^0)$. Choose w such that $f'(x^0)w + f''(x^0)(u) \in -\mathrm{int}\, P$ and observe that for w' in a suitable neighborhood of w it holds $f'(x^0)w' + f''(x^0)(u) \in -\mathrm{int}\, P$. If we assume that ii) does not hold, then we can find sequences $\delta_k \to 0$, $w^k \to w$, such that $f(x^0 + \delta_k u + \frac{\delta_k^2}{2} w^k) - f(x^0) \notin -\mathrm{int}\, P$. From Taylor's formula we get

$$\begin{aligned} f(x^0 + \delta_k u + \frac{\delta_k^2}{2} w^k) - f(x^0) &= \delta_k f'(x^0)u + \frac{\delta_k^2}{2}(f'(x^0)w^k + f''(x^0)(u)) + o(\|\delta_k u + \frac{\delta_k^2}{2} w^k\|) \\ &= \frac{\delta_k^2}{2}(f'(x^0)w^k + f''(x^0)(u)) + o(\|\delta_k u + \frac{\delta_k^2}{2} w^k\|)\,. \end{aligned}$$

Hence we obtain

$$\frac{\delta_k^2}{2}(f'(x^0)w^k + f''(x^0)(u)) + o(\|\delta_k u + \frac{\delta_k^2}{2} w^k\|) \notin -\mathrm{int}\, P\,.$$

Passing to the limit as $k \to +\infty$, we obtain the contradiction $(f'(x^0)w + f''(x^0)(u)) \notin -\mathrm{int}\, P$.

□

Remark 5.1. *From the definition of the index it follows in particular the existence of an $i^-(x^0)$-dimensional subspace L of $C^-_{f,P}(x^0)$ such that for every $u \in L$, i) and ii) of Theorem 5.1 hold. We underline that $C^-_{f,P}(x^0)$ is not invariant with respect to local coordinate changes in the domain and the image space. On the contrary, the dimension $i^-(x^0)$ is invariant with respcet to such local coordinate changes.*

Remark 5.2. *Statement i) in the previous theorem can be improved. Indeed, we can prove the existence of a positive number ε such that $f(x^0 + \delta u') - f(x^0) \in -\mathrm{int}\,(P + \mathrm{im}\, f'(x^0))$, for every $\delta \in (-\varepsilon, \varepsilon)$ and $u' \in u + \varepsilon B$. The proof follows an argument similar to the proof of Theorem 5.1, observing that the polar cone of $(P + \mathrm{im}\, f'(x^0))$ coincides with $\Lambda_{f,P}(x^0)$.*

With obvious modifications the previous results hold when we assume $u \in C^+_{f,P}(x^0)$.
In order to characterize the solutions of problem (1) through the index $I(x^0)$, we focus on the subset of $K(f,P)$ given by

$$\tilde{K}(f,P) = \{x \in K(f,P) : \mathrm{im}\, f'(x) \cap (-P) = \{0\}\}\,. \tag{9}$$

This restriction has already been considered by Smale [23]. In the sequel we characterize through the index $I(x^0)$ those points in $\tilde{K}(f,P)$ given by the locally efficient solutions of problem (1). Preliminarly we have to describe the behaviour of function f around the critical point $x^0 \in \tilde{K}(f,P)$ in the directions $u \notin \ker f'(x^0)$.

Proposition 5.1. *Let $x^0 \in \tilde{K}(f,P)$. If $u \notin \ker f'(x^0)$, then there exists a positive number ε such that for every nonzero $\delta \in (-\varepsilon, \varepsilon)$ and for every $u' \in u + \varepsilon B$, we have $f(x^0 + \delta u') - f(x^0) \notin (-P) \cup P$.*

Proof. Since $x^0 \in \tilde{K}(f,P)$ and $u \notin \ker f'(x^0)$, we have $f'(x^0)u \notin (-P) \cup P$. Using Taylor's formula we get the thesis. $\square$

The next result characterizes the locally efficient points of problem (1).

Theorem 5.2. *i) If $x^0 \in \mathrm{Eff}\,(f,P)$, then $i^-(x^0) = 0$.*

ii) Let $x^0 \in \tilde{K}(f,P)$ and assume that the following condition holds

$$H_{\Lambda,f}(x^0)(u) \cap -\partial\mathbb{R}_+^{l(x^0)} = \emptyset\,,\ \forall u \neq 0\,. \tag{10}$$

If $i^-(x^0) = 0$, then $x^0 \in \mathrm{Eff}\,(f,P)$.

Proof. i) The first statement follows from Theorem 5.1 ii).

ii) Let $u \notin \ker f'(x^0)$. From Proposition 5.1, we can find a positive number $\varepsilon(u)$ such that $f(x^0 + \delta u') - f(x^0) \notin -P$ for every $\delta \in (0, \varepsilon(u))$ and $u' \in u + \varepsilon B$.
Now we consider the directions of $\ker f'(x^0)$. Let $u \in C^+_{f,P}(x^0)$, $u \neq 0$. Then from Theorem 5.1, stated with respect to the directions of $C^+_{f,P}(x^0)$, we obtain again the existence of a positive number $\varepsilon(u)$ such that $f(x^0 + \delta u') - f(x^0) \notin -P$ for every $\delta \in (0, \varepsilon(u))$ and $u' \in u + \varepsilon(u)B$.
Finally, if $u \in C^{\pm}_{f,P}(x^0)$, using condition (10), we obtain the existence of a vector $\lambda \in \Lambda_{f,P}(x^0)$, such that $\lambda^T f''(x^0)(u) > 0$. Using Taylor's formula for function $\lambda^T f$, we get again the existence of a positive number $\varepsilon(u)$ such that $f(x^0+\delta u') - f(x^0) \notin -P$ for every $\delta \in (0, \varepsilon(u))$ and $u' \in u + \varepsilon(u)B$.
Hence, for every $u \in \mathbb{R}^n$, we have proved the existence of a positive number $\varepsilon(u)$ such that $f(x^0 + \delta u') - f(x^0) \notin -P$ for every $\delta \in (0, \varepsilon(u))$ and $u' \in u + \varepsilon(u)B$. This relation holds in particular for all the directions $u \in S$ (where S denotes the unit sphere in $\mathbb{R}^n$). Since S is compact we can find r directions $u^1, \ldots, u^r$ in S such that $\cup_{i=1}^r \varepsilon(u^i)B \subseteq S$. If we set $\bar{\varepsilon} = \min\{\varepsilon(u^i), i = 1, \ldots, r\}$, we obtain $f(x) - f(x^0) \notin -P$, $\forall x \in x^0 + \bar{\varepsilon}B$, $x \neq x^0$ and the proof is complete.

$\square$

Remark 5.3. *Point i) of Theorem 5.2 holds even if we substitute* $\mathrm{Eff}\,(f,P)$ *with* $\mathrm{WEff}\,(f,P)$.

Condition (10) can be interpreted as a "nondegeneracy" condition, since it imposes some restrictions on the zeroes of the vector of quadratic forms $H_{\Lambda,f}(x^0)$.

The index $I(x^0)$ allows us to deal also with the ideal solutions of problem (1).

Proposition 5.2. *i) If $i^+(x^0) = n$, then $x^0 \in \mathrm{IEff}\,(f,P)$.*

ii) If $x^0 \in \mathrm{IEff}\,(f,P)$ and $H_{\Lambda,f}(x^0)(u) \cap \partial\mathbb{R}_+^{l(x^0)} = \emptyset$, for $u \neq 0$, then $i^+(x^0) = n$.

Proof. i) Since $i^+(x^0) = n$, then $\operatorname{im} f'(x^0) = \{0\}$. Moreover, by Remark 5.2, for every direction u of unitary norm, we get the existence of a positive number $\varepsilon(u)$ such that $f(x^0 + \delta u') - f(x^0) \in \operatorname{int} P$, $\forall \delta \in (0, \varepsilon(u))$ and $\forall u' \in u + \varepsilon(u)B$. From the compactness of the unit sphere in $\mathbb{R}^n$ the thesis follows anologously to the proof of Theorem 5.2.

ii) If $x^0 \in \operatorname{IEff}(f, P)$, then the $k(x^0) = n$. Moreover, by Theorem 5.1 ii), we get $i^-(x^0) = 0$. We prove now that also $i^\pm(x^0) = 0$. Since $H_{\Lambda,f}(x^0)(u) \cap \partial\mathbb{R}_+^{l(x^0)} = \emptyset$, for every $u \in C^\pm_{f,P}(x^0)$, there exists a vector $\lambda \in \Lambda_{f,P}(x^0)$ such that $\lambda^T f''(x^0)(u) < 0$. By Taylor's formula, we easily get the existence of a positive number $\varepsilon(u)$ such that $\lambda^T(f(x^0 + \delta u') - f(x^0)) < 0$ for every $\delta \in (0, \varepsilon(u))$ and $u' \in u + \varepsilon(u)B$, which gives the contradiction $f(x^0 + \delta u') - f(x^0) \notin P$. By Theorem 4.2 ii), it holds $i^+(x^0) = n$.

□

6 Proper critical points and stability properties

In the previous section we introduced the set

$$\tilde{K}(f, P) = \{x \in K(f, P) : \operatorname{im} f'(x) \cap (-P) = \{0\}\}$$

in order to study the relationships between the index $I_{f,P}(x^0)$ and the efficient solutions of problem (1). Here we study some properties of $\tilde{K}(f, P)$. The first result shows that a point $x \in \tilde{K}(f, P)$, is critical also with respect to a cone P_1 with $P \subseteq \operatorname{int} P_1$. This property recalls the notion of properly efficient point introduced by Henig (see, e.g., [6]). Hence we call the points $x \in \tilde{K}(f, P)$ *properly critical points*.

Proposition 6.1. *Let $x \in \tilde{K}(f, P)$. Then there exists a closed convex pointed polyhedral cone P_1 with the same number of generators of P and $P \subseteq \operatorname{int} P_1$ such that $x \in \tilde{K}(f, P_1)$.*

Proof. It is similar to the proof of Proposition 6.2 below and we omit it. □

There is a clear relation between properly critical points and properly efficient points in the sense of Henig, i.e. every properly efficient point is a properly critical point.

The remaining part of this section is devoted to show that the assumption $x \in \tilde{K}(f, P)$ is not so restrictive. Indeed, a critical point of f can be always be considered as a proper critical point when we slightly perturb the order structure in the image space or of the function f. If $x^0 \in K(f, P) \backslash \tilde{K}(f, P)$, then it is possible to consider a perturbed order structure in the image space $\mathbb{R}^m$ such that x^0 is a proper critical point with respect to the new order structure. Indeed, if one considers any closed convex polyhedral cone $\tilde{P}$ with $\tilde{P} \subset \operatorname{int} P$, then it is easily seen that $x^0 \in \tilde{K}(f, \tilde{P})$. In particular, one can show that the perturbed cone $\tilde{P}$ can be chosen with the same structure as the original cone P and arbitrarily close to P.

Proposition 6.2. *Let $x^0 \in K(f,P)\backslash\tilde{K}(f,P)$. Then there exists a sequence of polyhedral cones $P_n \subseteq \operatorname{int} P$, with P_n converging to P in the sense of Kuratowski, with the same number of generators as P, such that $x^0 \in \tilde{K}(f,P_n)$, for n large enough.*

Proof. Let $\{a^1,\dots,a^r\}$ be a set of generators of P and let $\{d^n\}$ be a sequence of elements of $\operatorname{int} P$, such that $d^n \to 0$. Consider the vectors $a^{i,n} = a^i + d^n$, $i = 1,\dots,r$ and let $P_n = \operatorname{co\,conv}\{a^{1,n},\dots,a^{r,n}\}$. Clearly, $P_n \subseteq \operatorname{int} P$. Moreover, the vectors $a^{i,n}$, $i = \{1,\dots,r\}$, are extreme vectors for P_n, for n large enough. In order to prove this fact, it is sufficient to show that for n large enough, $a^{i,n}$, $i = \{1,\dots,r\}$ are extreme points of $\operatorname{conv}\{a^{1,n},\dots,a^{r,n}\}$.
Indeed, otherwise, one can find an index $j \in \{1,\dots,r\}$ and a sequence n_k such that for every k, there exists a^{j,n_k}, such that $a^{j,n_k} = t_k z^{j,k} + (1-t_k)v^{j,k}$, where $z^{j,k}, v^{j,k} \in \operatorname{conv}\{a^{1,n_k},\dots,a^{r,n_k}\}$, $z^{j,k} \neq v^{j,k}$ and $t_k \in (0,1)$. Since $z^{j,k} \in \operatorname{conv}\{a^{1,n_k},\dots,a^{r,n_k}\}$, there exist r nonnegative numbers $\lambda_{1,k},\dots,\lambda_{r,k}$ with $\sum_{s=1}^r \lambda_{s,r} = 1$ such that $z^{j,k} = \sum_{s=1}^r \lambda_{s,k} a^s + d^{n_k}$. Analogously, there exist r nonnegative numbers $\lambda'_{1,k},\dots,\lambda'_{r,k}$ with $\sum_{s=1}^r \lambda'_{s,k} = 1$ such that $v^{j,k} = \sum_{s=1}^r \lambda'_{s,k} a^s + d^{n_k}$. Hence $a^j = \sum_{s=1}^r (t_k\lambda_{s,k} + (1-t_k)\lambda'_{s,k})a^s$. But, taking into account $z^{j,k} \neq v^{j,k}$, we obtain a contradiction to the assumption that $a^1,\dots,a^r$ are generators of P.
For the sake of brevity, we just sketch the conclusion of the proof. Let $A = \operatorname{conv}\{a^1,\dots a^r\}$ and $A_n = \operatorname{conv}\{a^{1,n},\dots a^{r,n}\}$ be two compact bases respectively for P and P_n. From the choice of A_n and A, it follows easily that $A_n \to A$ where the convergence is intended in the Hausdorff sense. This is is equivalent to the Kuratowski convergence of P_n to P. □

We can also characterize the points in $\tilde{K}(f,P)$ in terms of the vectors $\lambda \in \operatorname{int} P'$. This approach is similar to the well-known definition of positive properly efficient points (see e.g. [6]).

Proposition 6.3. *A point x belongs to $\tilde{K}(f,P)$ if and only if $\Lambda_{f,P}(x) \cap \operatorname{int} P' \neq \emptyset$.*

Proof. We know that $x \in \tilde{K}(f,P)$ if and only if $\operatorname{im} f'(x) \cap (-P) = \{0\}$. Then from Proposition 2.1.8 in [20] it follows that this holds if and only if there exists a vector $\lambda \in \operatorname{int} P'$, such that $\lambda^T f'(x) = 0$ and the proof is complete. □

In the next proposition we prove that it is possible to perturb the function f in order to transform its critical points into the proper critical points of a suitably perturbed function $\tilde{f}$.

Proposition 6.4. *Let $K(f,P) \neq \emptyset$. Then there exists a function $\tilde{f}$ such that $K(f,P) \subseteq \tilde{K}(\tilde{f},P)$. Further, for every bounded set $C \in \mathbb{R}^n$ and for every positive number ε, $\tilde{f}$ can be chosen such that $\|f - \tilde{f}\| < \varepsilon$, where $\|f\| = \sup_{x\in C}(\|f(x)\| + \|f'(x)\| + \|f''(x)\|)$.*

Proof. Let w be a fixed vector belonging to $\operatorname{int} P$ and choose $\beta \in -\operatorname{int} P'$, such that $\beta^T w = -1$. Consider the function

$$\tilde{f}(x) = f(x) + t(\beta^T f(x))w\,,$$

where $t \in (0,1)$.
For any vector $\gamma \in P'$ such that $\gamma^T w = 1$, it holds

$$\gamma^T \tilde{f}'(x) = \gamma^T f'(x) + t\gamma^T w(\beta^T f'(x)) = (\gamma^T + t\beta^T) f'(x)\,.$$

Let $x \in K(f,P)$. Then, there exists $\lambda \in P' \backslash \{0\}$ such that $\lambda^T f'(x) = 0$. Moreover, we can always choose λ such that $\lambda^T w = 1 - t$. Now let $\gamma = \lambda - t\beta$. We obtain $\gamma^T \tilde{f}'(x) = 0$ and $\gamma \in \operatorname{int} P'$. By Proposition 6.3 the thesis follows. The last part of the statement trvially follows from the definition of $\tilde{f}$.

□

7 The case $c(x^0) = 1$

In the case $c(x^0) = 1$, the subspace $\{\lambda \in \mathbb{R}^m : \lambda^T f'(x^0) = 0\}$ is one-dimensional. Hence, $l(x^0) = 1$ and $H_{\Lambda,f}(x^0)(u)$ reduces to a quadratic form restricted to $\ker f'(x^0)$. For this reason, in the case $c(x^0) = 1$, it is not essential to restrict to polyhedral ordering cones. Indeed, since the set $\{\lambda \in \mathbb{R}^m : \lambda^T f'(x^0) = 0\}$ is a one-dimensional subspace of $\mathbb{R}^m$, then the index $I_{f,P}(x^0)$ is well-defined for an arbitary closed convex cone.
Moreover, we point out that the assumption $c(x^0) = 1$ is not excessively restrictive, as observed in [2], [21].

In the previous section we have shown that proper critical points enjoy some stability properties with respect to perturbations of the order structure and of the function. We have also seen that when a non-proper critical point x^0 is considered, then it is possible to build a perturbed problem (see Propositions 6.2 and 6.4) such that x^0 becomes a proper critical point. For our purposes, it is a fundamental issue to investigate whether the index of a critical point is invariant to such perturbations.

In the special case $c(x^0) = 1$ it is possible to show that $I_{f,P}(x^0)$ is invariant to the perturbations introduced in Proposition 6.2.

Proposition 7.1. *Let $x^0 \in K(f,P)$ with $c(x^0) = 1$. If $\tilde{P} \subset \operatorname{int} P$ is a closed convex pointed polyhedral cone with $\operatorname{int} \tilde{P} \neq \emptyset$, then $x^0 \in \tilde{K}(f,\tilde{P})$ and $I_{f,P}(x^0) = I_{f,\tilde{P}}(x^0)$.*

Proof. Let $x^0 \in K(f,P)$. If $x^0 \in \tilde{K}(f,P)$, from $\tilde{P} \subset \operatorname{int} P$ immediately follows $x^0 \in \tilde{K}(f,\tilde{P})$. If $x^0 \notin \tilde{K}(f,P)$, since $\operatorname{im} f'(x^0) \cap -\operatorname{int} P = \emptyset$, then we obtain again $x^0 \in \tilde{K}(f,\tilde{P})$. From the assumption $c(x^0) = 1$ it follows that the set $\{\lambda \in \mathbb{R}^m : \lambda^T f'(x^0) = 0\}$ is a one-dimensional subspace of $\mathbb{R}^m$. Since $P' \subset \tilde{P}'$ and $\Lambda_{f,\tilde{P}}(x^0) = \{\lambda \in \tilde{P}' : \lambda^T f'(x^0) = 0\}$, then it is easily seen that $\Lambda_{f,P}(x^0) = \Lambda_{f,\tilde{P}}(x^0)$ and hence the index is invariant.

□

In order to take into account the perturbations considered in Proposition 6.4 we state and prove the following auxiliary result. It is interesting in itself, since it shows that in a neighborhood of a critical point x^0 (satisfying an appropriate nondegeneracy condition) all the critical points have the same index.

Theorem 7.1. *Let $x^0 \in K(f,P)$ and $c(x^0) = 1$. If the quadratic form $\lambda^T f''(x^0)$, $\lambda \in \Lambda_{f,P}(x^0)$ with $\|\lambda\| = 1$ is nondegenerate when restricted to $\ker f'(x^0)$, then there exists a neighborhood U of x^0 such that, for every $x \in U \cap K(f,P)$, $I_{f,P}(x) = I_{f,P}(x^0)$.*

Proof. Since $c(x^0) = 1$ and f is of class C^2, there exists a neighborhood U of x^0 such that $c(x) = 1$, $\forall x \in U \cap K(f,P)$. Hence, $\dim\ker f'(x) = n - (m-1) = k(x^0)$, $\forall x \in U \cap K(f,P)$. We prove that if $x^s \to x^0$ as $s \to +\infty$, $x^s \in K(f,P)$ and $\lambda(x^s) \in \Lambda_{f,P}(x^s)$, $\|\lambda(x^s)\| = 1$, then (possibly extracting a subsequence) $\lambda(x^s) = \lambda^s \to \lambda \in \Lambda_{f,P}(x^0)$. Indeed, from $\|\lambda^s\| = 1$ and $\lambda^s \in P$ we can assume $\lambda^s \to \lambda \in P$, $\|\lambda\| = 1$ and from $\lambda^s f'(x^s) = 0$, it follows $\lambda f'(x^0) = 0$.
Now we prove that $\ker f'(x^s)$ converges to $\ker f'(x^0)$ in the sense of Kuratowski as $x^s \to x^0$. Since $x^0 \in K(f,P)$, then without loss of generality, we can write $f'_m(x^0) = \sum_{i=1}^{m-1} \lambda_i f'_i(x^0)$. Hence the points in $\ker f'(x^0)$ are the solutions of the following system of equations

$$\begin{cases} f'_1(x^0)u = 0 \\ \cdots \\ f'_{m-1}(x^0)u = 0 \end{cases}$$

The matrix

$$J = \begin{bmatrix} f'_1(x^0) \\ \cdots \\ f'_{m-1}(x^0) \end{bmatrix}$$

has full rank. From continuity arguments it follows that $\ker f'(x^s)$ is determined by the solutions of the system

$$\begin{cases} f'_1(x^s)u = 0 \\ \cdots \\ f'_{m-1}(x^s)u = 0 \end{cases}.$$

Hence, using Theorem 4.3.2 in [19] we have $\ker f'(x^s) \to \ker f'(x^0)$ in the sense of Kuratowski. Let now $(e^1,\dots,e^{k(x^0)})$ be an orthonormal base for $\ker f'(x^0)$. By the continuity of the inner product, we can find a sequence $\{(e^{1,s},\dots,e^{k(x^0),s})\}$ of bases of $\ker f'(x^s)$ such that $e^{j,s} \to e^j$, $\forall j = 1,\dots,k(x^0)$ as $s \to +\infty$. Let A_s be the matrix representing the quadratic form $(\lambda^s)^T f''(x^s)(\cdot)$ restricted to $\ker f'(x^s)$ (with respect to the base $(e^{1,s},\dots,e^{k(x^0),s})$) and let A be the matrix representing the quadratic form $\lambda^T f''(x^0)(\cdot)$ restricted to $\ker f'(x^0)$ (with respect to the base $(e^1,\dots,e^{k(x^0)})$). Direct calculations show that $A_s \to A$. Since $\lambda^T f''(x^0)(\cdot)$ is nondegenerate, all the principal minors of A are not zero. Since $c(x^0) = 1$, the index $I_{f,P}(x^0)$ is the triple (coindex, index, nullity) of the quadratic form $\lambda^T f''(x^0)(\cdot)$ restricted to $\ker f'(x^0)$.

Since this quadratic form is nondegenerate on $\ker f'(x^0)$, the nullity is 0 and the index $I_{f,P}(x^0)$ is completely determined by the index of the quadratic form $\lambda^T f''(x^0)(\cdot)$, i.e. by $i^-(x^0)$. Since $A_s \to A$, for s large enough, the sign of the principal minors of A_s coincides with the sign of the principal minors of A and this completes the proof. □

We conclude this section proving that under the assumption $c(x^0) = 1$, we can find a perturbation $\tilde{f}$ of the function f (in the sense of Proposition 6.4) such that the index of a critical point is invariant.

Theorem 7.2. *Let $x^0 \in K(f,P)$ and $c(x^0) = 1$. If the quadratic form $\lambda^T f''(x^0)$, $\lambda \in \Lambda(x^0)$ with $\|\lambda\| = 1$ is nondegenerate when restricted to $\ker f'(x^0)$, then there exists a neighborhood U of x^0 and a positive number $\bar{t}$ such that for every $x \in U \cap K(f,P)$ and every $t \in (0,\bar{t})$, the index of x as a critical point of f coincides with the index of x as a critical point of $\tilde{f}(x) = f(x) + t(\beta^T f(x))w$, where $w \in \operatorname{int} P$ and $\beta \in \operatorname{int} P'$ with $\beta^T w = 1$.*

Proof. First we prove that for every $x \in K(f,P)$, $\ker f'(x) = \ker \tilde{f}'(x)$. the inclusion $\ker f'(x) \subseteq \ker \tilde{f}'(x)$ trivially holds. Conversely, ab absurdo, assume $u \in \ker \tilde{f}'(x)$ and $u \notin \ker f'(x)$. From $u \in \ker \tilde{f}'(x)$ we get $f'(x)u = -t(\beta^T f'(x)u)w$ and $f'(x)u \neq 0$. We cannot have $\beta^T f'(x)u = 0$. Let, without loss of generality, $\beta^T f'(x)u > 0$. From $w \in \operatorname{int} P$, we obtain $f'(x)u \in -\operatorname{int} P$, a contradiction to $x \in K(f,P)$. Since $K(f,P) \subseteq K(\tilde{f},P)$ and f is of class C^2, there exists a neighborhood V of x^0 and a number $\tilde{t} \in (0,1)$ such that $\operatorname{corank} \tilde{f}'(x) = 1$, for every $x \in U \cap K(f,P)$ and $t \in (0,\tilde{t})$. Let $\tilde{\lambda} \in \Lambda_{\tilde{f},P}(x)$, $\|\lambda\| = 1$ and consider the quadratic form $\tilde{\lambda}\tilde{f}''(x)$. Since the quadratic form $\lambda^T f''(x^0)$ is nondegenerate, one can easily prove (following arguments similar to the proof of Theorem 7.1) the existence of a neighborhood U of x^0 and a positive number $\bar{t}$ such that for $x \in U$ and $t \in (0,\bar{t})$ the quadratic form $\tilde{\lambda}\tilde{f}''(x)$ is nondegenerate and has the same index of the quadratic form $\lambda f''(x)$, with $\lambda \in \Lambda_{f,P}(x)$ (both restricted to $\ker f'(x)$). The proof is complete using the previous theorem.

□

8 The case of maps from the plane to the plane

In the particular case $f : \mathbb{R}^2 \to \mathbb{R}^2$, the assumption $c(x^0) = 1$ is generic. This case has been studied by Whitney [27], who restricts his attention to the class of stable maps, which are dense in the space of C^∞ functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ endowed with the Whitney topology. Whitney proves that a stable map has only two kinds of singular points, namely cusps and folds (for a detailed and modern account on the topic see e.g. [1]). The aim of this section is to show that our approach essentially recovers the above mentioned classification. For the sake of brevity, we do not give here the definition of stable map. We just recall that a stable map is essentially a map which "preserves its form" under smooth changes of the independent and dependent variables. The interested reader can refer to [1].

Theorem 8.1. *[1] A function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is stable at a point if and only there exist two local coordinate systems in the domain and the image space such that the map can be represented as*

i) $y_1 = x_1, \; y_2 = x_2$ *(a regular point);*

ii) $y_1 = x_1^2, \; y_2 = x_2$ *(a fold);*

iii) $y_1 = x_1^3 + x_1 x_2, \; y_2 = x_2$ *(a cusp)*

(the point under consideration has the coordinates $x_1 = x_2 = 0$).

Now we characterize folds and cusps through the index $I_{f,P}(x^0)$. The key tool used in the proof is the invariance on the index both in the domain and the image space (see Proposition 4.1).

Proposition 8.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a stable map. A point $x^0 \in \mathbb{R}^2$ is a singular point of f if and only if there exists a closed convex pointed cone $\tilde{P} \subseteq \mathbb{R}^2$ with $\operatorname{int} \tilde{P} \neq \emptyset$ such that $x^0 \in K(f, \tilde{P})$. Further x^0 is fold (a cusp) if and only if for such a cone it holds $I_{f,\tilde{P}}(x^0) = (1, 0, 0)$ ($I_{f,\tilde{P}}(x^0) = (0, 0, 1)$).*

Proof. Without loss of generality we can assume $x^0 = 0$. Clearly, if $x^0 \in K(f, P)$ for some cone P, then x^0 is singular. Conversely, let x^0 be a singular point. Then, one can find two local coordinate changes h and g in the image and the domain space, respectively, such that f can be described in the normal forms ii) or iii) of Theorem 8.1. It is easy to see, by direct calculations with respect to these systems of coordinates, $x^0 \in K(f, P)$ for every closed convex pointed cone P with $\operatorname{int} P \neq \emptyset$ and $P \subseteq C\{(y_1, y_2) \in \mathbb{R}^2 : y_1 \geq 0\}$. For one of such P, considering $\tilde{P} = \{[h'(f(x^0))]^T\}^{-1} P$ one gets the desired cone in the original coordinate systems.
If local coordinate systems are introduced so that f is given in one of the forms ii) or iii) of Theorem 8.1, then it is easy to see that x^0 is a fold if and only if $I_{f,P}(x^0) = (1, 0, 0)$ for every cone $P \subset C$, while x^0 is a cusp if and only if $I_{f,P}(x^0) = (0, 0, 1)$ for every $P \subseteq C$. Taking again $\tilde{P} = \{[h'(f(x^0))]^T\}^{-1} P$, when we consider the original coordinate systems, then x^0 is a fold if and only if $I_{f,\tilde{P}}(x^0) = (1, 0, 0)$, while x^0 is a cups if and only if $I_{f,\tilde{P}}(x^0) = (0, 0, 1)$. □

We underline that in the case of stable maps from the plane into the plane, the index $I_{f,P}(x^0)$ can assume only three possible values, that is $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ (see Theorem 4.2, with $k(x^0) = 1$). In the classical approach by Whitney, the fold points correspond to the values of the index $(1, 0, 0)$ and $(0, 1, 0)$ which can be obtained taking as ordering cones repectively P and $-P$. In our approach, since we consider the image space $\mathbb{R}^2$ as an ordered space, these two values of the index identify distinct situations.

When we consider maps from $\mathbb{R}^n \to \mathbb{R}^m$ with arbitrary $n \geq m$, the study of singular points is more complicate since stable maps are not necessarily dense in the space of C^∞ functions from

$\mathbb{R}^n$ to $\mathbb{R}^m$ (see, e.g., [1]). Moreover, even in the case of stable maps, the number of different kinds of singularities grows up. In particular one can extend the notion of fold point and give a canonical form. Indeed, there exist local coordinate changes in the domain and the image space such that the function f can be represented as

$$f_i(x) = x_i\,, \quad i = 1, \ldots, m-1$$

$$f_i(x) = \pm x_m^2 \pm x_{m+1}^2 \pm \cdots \pm x_n^2$$

in a neighborhood of a fold point. Clearly, when x^0 is a fold point, then $c(x^0) = 1$.
If we restrict our attention to maps from $\mathbb{R}^n \to \mathbb{R}^n$, we can generalize the characterization of fold points given in Proposition 8.1. Namely we can prove analogously that a point x^0 with $c(x^0) = 1$ is a fold if and only if there exists a closed convex pointed cone $\tilde{P} \subseteq \mathbb{R}^n$ with $\operatorname{int} \tilde{P} \neq \emptyset$ such that $I_{f,\tilde{P}}(x^0) = (1, 0, 0)$.

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