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Pareto reducibility of vector variational inequalities*

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Abstract

A multicriteria optimization problem is called Pareto reducible if its weakly efficient solutions are Pareto solutions of the problem itself or a subproblem obtained from it by selecting certain criteria. The aim of this paper is to introduce a similar concept of Pareto reducibility for a class of vector variational inequalities.

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1 Multicriteria optimization problems

Consider a vector function, $f = (f_1, \ldots, f_m) : D \to \mathbb{R}^m$, defined on a nonempty set D and taking values in the m-dimensional real Euclidean space $\mathbb{R}^m$ ($m \in \mathbb{N}$, $m \geq 2$).

For convenience, denote $K_m := \{1, \ldots, m\}$. Whenever $K \subset K_m$ will be a nonempty set of indices, with cardinality $|K| = k$, the notation f_K will represent the function $f_K = (f_{i_1}, \ldots, f_{i_k}) : D \to \mathbb{R}^k$, where $i_1, \ldots, i_k$ are implicitly defined by

$$K = \{i_1, \ldots, i_k\} \quad \text{and} \quad i_1 < \ldots < i_k.$$

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To each such a selection of indices, we associate an optimization problem:

$$(\mathcal{P}_K) \quad \begin{cases} \text{Minimize} & f_K(x) \\ \text{subject to} & x \in D. \end{cases}$$

Notice that $(\mathcal{P}_K)$ is a scalar optimization problem if K is a singleton, otherwise being a vector optimization problem.

As usual in Vector Optimization (see, e.g., [7]), the set of *Pareto solutions* (also called *efficient solutions*) of $(\mathcal{P}_K)$ is given by

$$\mathrm{Eff}(\mathcal{P}_K) \;:=\; \{x^0 \in D \;\mid\; \nexists\, x \in D : f_K(x) \in f_K(x^0) - \mathbb{R}^k_+ \setminus \{0\}\}$$

while that of *weakly efficient solutions* is defined as

$$\text{w-Eff}(\mathcal{P}_K) \;:=\; \{x^0 \in D \;\mid\; \nexists\, x \in D : f_K(x) \in f_K(x^0) - \operatorname{int} \mathbb{R}^k_+\}.$$

Since $f_{K_m} = f$, it follows that whenever $K \neq K_m$ the optimization problem $(\mathcal{P}_K)$ can be regarded as a subproblem (i.e., a reduced problem) obtained from

$$(\mathcal{P}_{K_m}) \quad \begin{cases} \text{Minimize} & f(x) \\ \text{subject to} & x \in D \end{cases}$$

by eliminating certain criteria.

It is easily seen that, for every nonempty subset K of K_m, one has

$$\mathrm{Eff}(\mathcal{P}_K) \subset \text{w-Eff}(\mathcal{P}_K) \subset \text{w-Eff}(\mathcal{P}_{K_m}). \tag{1}$$

According to Popovici [8], the multicriteria optimization problem $(\mathcal{P}_{K_m})$ is said to be Pareto reducible if

$$\text{w-Eff}(\mathcal{P}_{K_m}) = \bigcup_{\emptyset \neq K \subset K_m} \mathrm{Eff}(\mathcal{P}_K), \tag{2}$$

which means that every weakly efficient solution of $(\mathcal{P}_{K_m})$ actually is a Pareto solution of $(\mathcal{P}_{K_m})$ or it is a Pareto solution for at least one subproblem of type $(\mathcal{P}_K)$, where K is a proper subset of K_m.

Obviously, the inclusion "$\supset$" in (2) is always true, as shown by (1). However, in order to establish the inclusion "$\subset$" in (2), some additional assumption have to be imposed on the objective function f.

Motivated by the practical importance of location problems, the relation (2) has been proven by Lowe et al. in [6] by assuming that all scalar components of f are convex, in this case the proof being based on the classical weighting scalarization method. Since this scalarization method cannot be applied for general nonconvex problems, some other appropriate generalized convexity assumptions have been identified in [9], [10], and [4], in order to obtain sufficient conditions for the Pareto reducibility of the multicriteria optimization problem $(\mathcal{P}_{K_m})$.

2 Vector variational inequalities

Throughout this section D will be a nonempty closed convex subset of the n-dimensional real Euclidean space $\mathbb{R}^n$, which will be endowed with the usual inner product $\langle \cdot, \cdot \rangle$.

Let $F_1 : D \to \mathbb{R}^n, \ldots, F_m : D \to \mathbb{R}^n$ some vector-valued functions ($m \in \mathbb{N}$, $m \geq 2$). For all $x \in D$ and $y \in \mathbb{R}^n$, denote

$$F(x)(v) \ := \ (\langle F_1(x), y \rangle, \ldots, \langle F_m(x), y \rangle) \in \mathbb{R}^m \tag{3}$$

and consider the following vector variational inequalities (see [2] and [1], respectively):

$$\text{(VVI)} \qquad \text{Find } \bar{x} \in D \text{ such that } F(\bar{x})(x - \bar{x}) \notin -\mathbb{R}^m_+ \setminus \{0\}, \ \forall\, x \in D;$$

$$\text{(w-VVI)} \qquad \text{Find } \bar{x} \in D \text{ such that } F(\bar{x})(x - \bar{x}) \notin -\operatorname{int} \mathbb{R}^m_+, \ \forall\, x \in D.$$

Let us denote by $\operatorname{Sol}(\text{VVI}) := \{\bar{x} \in D \mid F(\bar{x})(x - \bar{x}) \notin -\mathbb{R}^m_+ \setminus \{0\}, \ \forall\, x \in D\}$ and $\operatorname{Sol}(\text{w-VVI}) := \{\bar{x} \in D \mid F(\bar{x})(x - \bar{x}) \notin -\operatorname{int} \mathbb{R}^m_+, \ \forall\, x \in D\}$ the solution sets of the above vector variational inequalities.

Following the scalarization approach proposed by Lee et al. in [5], we will associate to each vector $\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m_+$ the following variational inequality:

$$(\text{VI}_\xi) \qquad \text{Find } \bar{x} \in D \text{ such that } \langle \textstyle\sum_{i=1}^m \xi_i F_i(\bar{x}), x - \bar{x} \rangle \geq 0, \ \forall\, x \in D.$$

Notice that (VI_ξ) is a variational inequality in the classical sense, as considered in [3]. Let $\operatorname{Sol}(\text{VI}_\xi) := \{\bar{x} \in D \mid \langle \sum_{i=1}^m \xi_i F_i(\bar{x}), x - \bar{x} \rangle \geq 0, \ \forall\, x \in D\}$ be the solution set of (VI_ξ).

The following preliminary result has been established by Lee et al. ([5], Theorem 2.1).

Lemma 1 *The following relations hold true*:

$$\bigcup_{\xi \in \operatorname{int} \mathbb{R}^m_+} \operatorname{Sol}(\text{VI}_\xi) \subset \operatorname{Sol}(\text{VVI}) \subset \operatorname{Sol}(\text{w-VVI}) = \bigcup_{\xi \in \mathbb{R}^m_+ \setminus \{0\}} \operatorname{Sol}(\text{VI}_\xi). \tag{4}$$

As we have seen in Section 1, a multicriteria optimization problem can be decomposed into subproblems obtained from the original one by eliminating certain criteria. Similarly, we will associate to each nonempty set of indices $K = \{i_1, \ldots, i_k\} \subset K_m := \{1, \ldots, m\}$ (where it is implicitly understood that $i_1 < \ldots < i_k$) two variational inequalities, which can be viewed as subproblems of the vector variational inequalities (VVI) and (w-VVI). To this aim, we define for all $x \in D$ and $y \in \mathbb{R}^n$ the following point in $\mathbb{R}^k$:

$$F_K(x)(y) \ := \ (\langle F_{i_1}(x), y \rangle, \ldots, \langle F_{i_k}(x), y \rangle). \tag{5}$$

Now, we can introduce the following two variational inequalities:

$$(\text{VVI}_K) \qquad \text{Find } \bar{x} \in D \text{ such that } F_K(\bar{x})(x - \bar{x}) \notin -\mathbb{R}^k_+ \setminus \{0\}, \ \forall\, x \in D;$$

$$(\text{w-VVI}_K) \qquad \text{Find } \bar{x} \in D \text{ such that } F_K(\bar{x})(x - \bar{x}) \notin -\operatorname{int} \mathbb{R}^k_+, \ \forall\, x \in D.$$

The sets of their solutions will be denoted by $\operatorname{Sol}(\text{VVI}_K)$ and $\operatorname{Sol}(\text{w-VVI}_K)$, respectively.

Remark 1 When the cardinality of K is greater than one (VVI_K) and (w-VVI_K) actually are vector variational inequalities; otherwise, if K is a singleton, they become classical (i.e. scalar) variational inequalities. Notice also that for $K = K_m$ we recover the original vector variational inequalities and we have

$$\mathrm{Sol}(\mathrm{VVI}_{K_m}) = \mathrm{Sol}(\mathrm{VVI}) \quad (6)$$

$$\mathrm{Sol}(\text{w-}\mathrm{VVI}_{K_m}) = \mathrm{Sol}(\text{w-}\mathrm{VVI}). \quad (7)$$

Theorem 1 *The following equality holds:*

$$\mathrm{Sol}(\text{w-}\mathrm{VVI}) = \bigcup_{\emptyset \neq K \subset K_m} \mathrm{Sol}(\mathrm{VVI}_K). \quad (8)$$

Proof. We will firstly prove the inclusion "$\subset$" in (8). Consider an arbitrary point $\tilde{x} \in \mathrm{Sol}(\text{w-}\mathrm{VVI})$. By (4) we infer the existence of a vector $\tilde{\xi} = (\tilde{\xi}_1, \ldots, \tilde{\xi}_m) \in \mathbb{R}^m_+ \setminus \{0\}$ such that $\tilde{x} \in \mathrm{Sol}(\mathrm{VI}_{\tilde{\xi}})$, i.e.,

$$\langle \textstyle\sum_{i=1}^m \tilde{\xi}_i F_i(\tilde{x}), x - \tilde{x} \rangle \geq 0, \ \forall\, x \in D. \quad (9)$$

Consider the set $\tilde{K} := \{i \in K_m \mid \tilde{\xi} > 0\}$. This set is nonempty, since $\tilde{\xi} \in \mathbb{R}^m_+ \setminus \{0\}$. Denoting by k the cardinality of $\tilde{K}$, it follows that $\tilde{K} = \{i_1, \ldots, i_k\}$ for some indices $i_1 < \ldots < i_k$ from K_m.

Consider the vector $\tilde{\eta} = (\tilde{\eta}_1, \ldots, \tilde{\eta}_k) \in \mathrm{int}\, \mathbb{R}^k_+$, defined by

$$\tilde{\eta}_j := \tilde{\xi}_{i_j} \quad \text{for all} \quad j \in \{1, \ldots, k\}.$$

Then (9) can be rewritten as

$$\langle \textstyle\sum_{j=1}^k \tilde{\eta}_j F_{i_j}(\tilde{x}), x - \tilde{x} \rangle \geq 0, \ \forall\, x \in D,$$

which shows that $\tilde{x} \in \mathrm{Sol}(\mathrm{VI}_{\tilde{K}, \tilde{\eta}}) := \{\bar{x} \in D \mid \langle \sum_{j=1}^k \tilde{\eta}_j F_{i_j}(\bar{x}), x - \bar{x} \rangle \geq 0, \ \forall\, x \in D\}$.

By applying Lemma 1 for $(F_{i_1}, \ldots, F_{i_k})$ instead of $(F_1, \ldots, F_m)$, we obtain that

$$\bigcup_{\eta \in \mathrm{int}\, \mathbb{R}^k_+} \mathrm{Sol}(\mathrm{VI}_{\tilde{K}, \eta}) \subset \mathrm{Sol}(\mathrm{VVI}_{\tilde{K}}) \subset \mathrm{Sol}(\text{w-}\mathrm{VVI}_{\tilde{K}}) = \bigcup_{\eta \in \mathbb{R}^k_+ \setminus \{0\}} \mathrm{Sol}(\mathrm{VI}_{\tilde{K}, \eta}), \quad (10)$$

where $\mathrm{Sol}(\mathrm{VI}_{\tilde{K}, \eta}) := \{\bar{x} \in D \mid \langle \sum_{j=1}^k \eta_j F_{i_j}(\bar{x}), x - \bar{x} \rangle \geq 0, \ \forall\, x \in D\}$ for any vector $\eta = (\eta_1, \ldots, \eta_k) \in \mathbb{R}^k_+$.

Since $\tilde{x} \in \mathrm{Sol}(\mathrm{VI}_{\tilde{K}, \tilde{\eta}}) \subset \bigcup_{\eta \in \mathrm{int}\, \mathbb{R}^k_+} \mathrm{Sol}(\mathrm{VI}_{\tilde{K}, \eta})$, it follows from (10) that $\tilde{x} \in \mathrm{Sol}(\mathrm{VVI}_{\tilde{K}})$, hence $\tilde{x} \in \bigcup_{\emptyset \neq K \subset K_m} \mathrm{Sol}(\mathrm{VVI}_K)$. Thus the inclusion "$\subset$" in (8) is true.

In order to prove the inclusion "$\supset$" in (8), let $\hat{x} \in \bigcup_{\emptyset \neq K \subset K_m} \mathrm{Sol}(\mathrm{VVI}_K)$ be arbitrarily chosen. Then there exists a set of indices $K = \{i_1, \ldots, i_k\} \subset K_m$ such that $i_1 < \ldots < i_k$ and $\hat{x} \in \mathrm{Sol}(\mathrm{VVI}_K)$.

By applying Lemma 1 for $(F_{i_1}, \dots, F_{i_k})$ in the role of $(F_1, \dots, F_m)$, we obtain that

$$\bigcup_{\eta \in \operatorname{int} \mathbb{R}^k_+} \mathrm{Sol}(\mathrm{VI}_{K,\eta}) \subset \mathrm{Sol}(\mathrm{VVI}_K) \subset \mathrm{Sol}(\text{w-VVI}_K) = \bigcup_{\eta \in \mathbb{R}^k_+ \setminus \{0\}} \mathrm{Sol}(\mathrm{VI}_{K,\eta}), \tag{11}$$

where $\mathrm{Sol}(\mathrm{VI}_{K,\eta}) := \{\bar{x} \in D \mid \langle \sum_{j=1}^k \eta_j F_{i_j}(\bar{x}), x - \bar{x}\rangle \geq 0,\ \forall x \in D\}$ for any vector $\eta = (\eta_1, \dots, \eta_k) \in \mathbb{R}^k_+$.

Since $\hat{x} \in \mathrm{Sol}(\mathrm{VVI}_K)$, it follows by (11) that $\hat{x} \in \bigcup_{\eta \in \mathbb{R}^k_+ \setminus \{0\}} \mathrm{Sol}(\mathrm{VI}_{K,\eta})$, i.e., there exists $\hat{\eta} = (\hat{\eta}_1, \dots, \hat{\eta}_k) \in \mathbb{R}^k_+ \setminus \{0\}$ such that

$$\langle \sum_{j=1}^k \hat{\eta}_j F_{i_j}(\hat{x}), x - \hat{x}\rangle \geq 0,\ \forall x \in D. \tag{12}$$

Taking into account that $K = \{i \in K_m \mid \exists j \in \{1, \dots, k\} \text{ such that } i = i_j\}$ and $i_1 < \dots < i_k$, we can define a vector $\hat{\xi} = (\hat{\xi}_1, \dots, \hat{\xi}_m) \in \mathbb{R}^m$ by

$$\hat{\xi}_i := \begin{cases} \hat{\eta}_i & \text{if} \quad i = i_j \text{ for some } j \in \{1, \dots, k\} \\ 0 & \text{if} \quad i \in K_m \setminus K. \end{cases}$$

Then we have $\sum_{i=1}^m \hat{\xi}_i F_i(\hat{x}) = \sum_{j=1}^k \hat{\eta}_j F_{i_j}(\hat{x})$. Hence (12) becomes

$$\langle \sum_{i=1}^m \hat{\xi}_i F_i(\hat{x}), x - \hat{x}\rangle \geq 0,\ \forall x \in D,$$

which means that $\hat{x} \in \mathrm{Sol}(\mathrm{VI}_{\hat{\xi}})$. On the other hand, it is easily seen that $\hat{\xi} \in \mathbb{R}^m \setminus \{0\}$, since $\hat{\eta} \in \mathbb{R}^k_+ \setminus \{0\}$. By (4) we infer that $\hat{x} \in \mathrm{Sol}(\text{w-VVI})$. Thus the inclusion "$\supset$" in (8) is true. $\square$

Remark 2 Theorem 1 shows that the vector variational inequalities are Pareto reducible, in the sense that every solution of the vector variational inequality Sol(w-VVI) is a solution of at least one "reduced" variational inequality of type $\mathrm{Sol}(\mathrm{VVI}_K)$, with $K \subset K_m$.

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